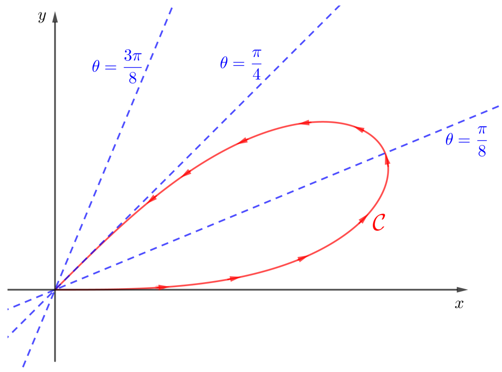
1092 Calculus4 06-12 Final Exam Solution

June 19, 2021

1. (12 pts) Use Green's theorem to evaluate the line integral

$$\oint_C (x^2 + y^3)\mathrm{d}x + (x^3 + y^2)\mathrm{d}y$$

where C traces counter-clockwisely the polar curve $r^2 = \sin(4\theta)$, $0 \le \theta \le \dfrac{\pi}{4}$.



Solution:

The following steps of arguments must be shown clearly!

(2%) State what is the Green's theorem:

$\oint_{\mathcal{C}}(x^2 + y^3)dx + (x^3 + y^2)dy = \int\int_{\mathcal{D}}[(x^3 + y^2)_x - (x^2 + y^3)_y]dxdy$
where $\mathcal{D}$ is the region enclosed by $r^2 = \sin(4\theta)$, $0 \le \theta \le \pi/4$.

(2%) Find the relevant partial derivatives and the link to polar coordinates:

$(x^3 + y^2)_x - (x^2 + y^3)_y = 3(x^2 - y^2) = 3r^2 \cos 2\theta$

(3%) Iteration for the double integral in terms of polar coordinates:

$\int\int_{\mathcal{D}}[(x^3 + y^2)_x - (x^2 + y^3)_y]dxdy = \int_0^{\pi/4}\int_0^{\sqrt{\sin(4\theta)}} 3r^3\cos(2\theta)drd\theta$

$= \frac{3}{4}\int_0^{\pi/4}\cos(2\theta)\sin^2(4\theta)d\theta$

(5%) Work out the trigonometric integration:

$= \frac{3}{8}\int_0^{\pi/4}\cos(2\theta)[1 - \cos(8\theta)]d\theta = \frac{3}{8}\int_0^{\pi/4}[\cos(2\theta) - \frac{1}{2}(\cos(6\theta) + \cos(10\theta))]d\theta$

$= \frac{3}{8}\left[\frac{\sin(2\theta)}{2} - \frac{\sin(6\theta)}{12} - \frac{\sin(10\theta)}{20}\right]_0^{\pi/4} = \frac{3}{8}\left[\frac{1}{2} + \frac{1}{12} - \frac{1}{20}\right] = \frac{1}{5}$

2. Let α and β be two constants and set

$$\mathbf{F}(x,y,z) = (y + \alpha z e^{xz})\,\mathbf{i} + \beta x\,\mathbf{j} - 2xe^{xz}\,\mathbf{k}.$$

(a) (5 pts) Find curl($\mathbf{F}$) in terms of α and β.

(b) (4 pts) Find the values of α and β such that $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

(c) (6 pts) For the pair of values that you found in (b), evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where

$$C = \{(x,y,z) \in \mathbb{R}^3 : z = 2x^2 = y^3,\ 0 \le x \le 2\}$$

oriented in increasing values of x.

Solution:

(a) (Total 5%) $\text{curl}(\mathbf{F}) = 0\mathbf{i} + (2+\alpha)(1+xz)e^{xz}\mathbf{j} + (\beta - 1)\mathbf{k}$ (5 pts)

Partial credit below is given if the answer is not completely correct:

- Write down the correct definition for curl($\mathbf{F}$) (2 pt)
- Write down correct component of **i,j,k** (1 pt+1 pt+1 pt)

(b) (Total 4%)

- State the correct statement: [F is conservative on $\mathbb{R}^3$ iff $curl\mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$] or [if $curl\mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$ then F is conservative on $\mathbb{R}^3$] (2 pts)
- $\alpha = -2$, $\beta = 1$ (2 pts)

(c) (Total 6%) Method 1: Find potential function and use fundamental theorem of line integral:

- Write down $\mathbf{F}(x,y,z) = (y - 2ze^{xz})\mathbf{i} + x\mathbf{j} - 2xe^{xz}\mathbf{k}$ or mention anywhere about the correct substitution of α and β into $\mathbf{F}$ (1 pt)
- Solving for potential function $\mathbf{F} = \nabla f$: $f(x,y,z) = xy - 2e^{xz}$ (2 pts, if students include any constant c behind f, it is acceptable)
- By Fundamental theorem of line integral,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,2,8) - f(0,0,0) = 6 - 2e^{16}$$

  (3 pts; 1 pt for correct initial point $(0,0,0)$, 1 pt for correct terminal point $(2,2,8)$, 1 pt for correct answer)

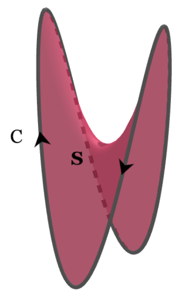
Method 2: Direct computation of line integral

- Write down $\mathbf{F}$ correctly or mention anywhere about the correct substitution of α and β into $\mathbf{F}$ (1 pt).
- Any correct parametrization AND orientation of C (1 pt).
- Correct computation of line integral. (3 pts).
- Correct answer $= 6 - 2e^{16}$ (1 pt).

Remark: For Method 2, if answer is incorrect, the maximum score is 2 pts. Only need to check for $\mathbf{F}$ and C.

3. The figure on the right gives a surface S which is part of the graph $z = y^2 - x^2$ enclosed by the curve C with parameterization

$$\mathbf{r}(t) = \langle 2\sin(t),\, 2\cos(t),\, 4\cos(2t)\rangle \text{ with } 0 \le t \le 2\pi.$$



(a) (7 pts) Find the surface area $\iint_S 1\,\mathrm{d}S$ of S.

(b) (8 pts) By Stokes' Theorem, evaluate the circulation

$$\oint_C (ye^x - y^3)\,\mathrm{d}x + (x^3 + e^x)\,\mathrm{d}y + (z^2e^z)\,\mathrm{d}z.$$

Solution:

Marking Scheme for Question 3a

- 1% - (1) Parametrisation of S
- 1% - (2) Correct specification of the ranges of parameters D
- 1% - (3) Correct value of $\|\mathbf{r}_x \times \mathbf{r}_y\|$
- 2% - (4) Correct definition of surface integrals (at most 1% for those who does not/cannot specify D or makes mistakes in $\|\mathbf{r}_\mathbf{x} \times \mathbf{r}_y\|$)
- 2% - (5) Correct evaluation (1% for students who made minor sign errors/obvious typos)

Remark. In other words, if a student makes mistakes in (2) or (3), he/she can earn at most 3% for this part of the question.

Sample Solution to Q3(a) Ver 1.

Parametrise S by $\mathbf{r}(x,y) = \underbrace{\langle x, y, y^2 - x^2\rangle}_{1\%}$ where $x, y \in \underbrace{D = \{(x,y) : x^2 + y^2 \le 4\}}_{1\%}$. Then

$$\iint_S 1\,\mathrm{d}S \overset{\text{def}}{=} \underbrace{\iint_D \underbrace{\sqrt{4x^2 + 4y^2 + 1}}_{1\%}\,dA}_{2\%} \overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta = \underbrace{\frac{\pi}{6}\left(17^{\frac{3}{2}} - 1\right)}_{2\%}.$$

Sample Solution to Q3(a) Ver 2.

Parametrise S by $\mathbf{r}(r,\theta) = \underbrace{\langle r\sin\theta, r\cos\theta, r^2\cos(2\theta)\rangle}_{1\%}$ where $\underbrace{0 \le r \le 2,\ 0 \le \theta \le 2\pi}_{1\%}$.

Then $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \underbrace{r\sqrt{4r^2+1}}_{1\%}$ and

$$\iint_S 1\,\mathrm{d}S \overset{\text{def}}{=} \underbrace{\int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta}_{2\%} = \underbrace{\frac{\pi}{6}\left(17^{\frac{3}{2}} - 1\right)}_{2\%}.$$

Marking Scheme for Question 3b

- 1% - (1) Statement of Stokes Theorem
- 1% - (2) Correct computation of curl($\mathbf{F}$)
- 1% - (3) Correct ($\mathbf{k}$-component of) $\mathbf{r}_y \times \mathbf{r}_x$
- 3% - (4) Correct transformation of a flux integral into a double integral (at most 1% for those who does not/cannot specify D in (a) or makes mistakes in (2) or (3))
- 2% - (5) Correct evaluation (1% for students who made a sign error)

Remark a. In other words, if a student makes mistakes in (2) or (3) or not specifying D, he/she can earn about 3-4% for this part of the question.
Remark b. Sign error will be deducted from (5) only.

Sample Solution to Q3(b) Ver 1.

By Stokes' Theorem, $\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}}_{1\%}$.

To evaluate the RHS, first we compute that $\underbrace{\operatorname{curl}(\mathbf{F}) = (3x^2+3y^2)\mathbf{k}}_{1\%}$ and therefore

$$\iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_D \langle 0,0,3x^2+3y^2\rangle \cdot \underbrace{\langle -2x, 2y, -1\rangle}_{1\%}\, dA = \underbrace{-3\iint_D (x^2+y^2)\, dA}_{3\%}$$

$$\overset{\text{Polar}}{=} -3\int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta$$

$$= \underbrace{-24\pi}_{2\%}$$

Sample Solution to Q3(b) Ver 2.

By Stokes' Theorem, $\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}}_{1\%}$.

To evaluate the RHS, first we compute that $\underbrace{\operatorname{curl}(\mathbf{F}) = (3x^2+3y^2)\mathbf{k}}_{1\%}$ and therefore

$$\iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 \langle 0,0,3r^2\rangle\cdot \underbrace{\langle \star,\star,-r\rangle}_{1\%}\, dr\, d\theta = \underbrace{-3\int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta}_{3\%} = \underbrace{-24\pi}_{2\%}$$

4. Consider the field of gravitational force $\mathbf{F}:\mathbb{R}^3\smallsetminus\{(0,0,0)\}\to\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z)=\frac{-1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

(a) (6 pts) Show that $\mathrm{div}(\mathbf{F}(x,y,z))=0$ for any $(x,y,z)\neq(0,0,0)$.

(b) (7 pts) Evaluate, directly, the flux of $\mathbf{F}$ across the sphere $S_R: x^2+y^2+z^2=R^2$ where $R>0$ endowed with outward orientation.

(c) (5 pts) Hence, find the flux of $\mathbf{F}$ across the ellipsoid $E: x^2+y^2+2z^2=1$ oriented outward.

Solution:

For (a), let $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$. The formula of divergence is

$$\mathrm{div}(\mathbf{F})=(F_1)_x+(F_2)_y+(F_3)_z. \tag{1}$$

It is straightforward to obtain

$$(F_1)_x(x,y,z)=\frac{-1+3x^2(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{\frac{3}{2}}}. \tag{2}$$

Remark: One can obtain $(F_2)_y$ and $(F_3)_z$ in a straightforward manner like (2), or just say that $\mathbf{F}$ is symmetric in the symbols x,y,z.

For (b), the flux is given by

$$\iint_{S_R}\mathbf{F}(x,y,z)\cdot\frac{(x,y,z)}{R}\,\mathrm{d}S. \tag{3}$$

Applying spherical coordinates yields

$$\iint_{S_R}\frac{-(x,y,z)}{R^3}\cdot\frac{(x,y,z)}{R}\,\mathrm{d}S=\frac{-1}{R^2}\iint_{S_R}\mathrm{d}S \tag{4}$$

$$=\frac{-1}{R^2}4\pi R^2 \tag{5}$$

$$=-4\pi. \tag{6}$$

Remark: One can use other methods than spherical coordinates to evaluate the flux.

For (c), let $R>0$ be sufficiently small so that S_R is contained in the interior of the ellipsoid E. We endow S_R with the orientation in the item (b). Let Ω_R be the space region between S_R and E.

Since $(0,0,0)$ is not in Ω_R, we can apply the divergence theorem on Ω_R and use the item (a) to obtain

$$\left(\iint_E-\iint_{S_R}\right)\mathbf{F}\cdot\mathrm{d}\vec{S}=\iiint_{\Omega_R}\mathrm{div}(\mathbf{F})\,\mathrm{d}V \tag{7}$$

$$=0. \tag{8}$$

Therefore, by the item (b) we see

$$\iint_E\mathbf{F}\cdot\mathrm{d}\vec{S}=\iint_{S_R}\mathbf{F}\cdot\mathrm{d}\vec{S} \tag{9}$$

$$=-4\pi. \tag{10}$$

Grading Suggestion.

- *For* (a), *get* 2/6 *by obtaining* (1); *get* 1/6 *by obtaining each* $(F_1)_x$, $(F_2)_y$, *and* $(F_3)_z$ *as shown similarly in* (2); *get* 1/6 *by explaining why* $\operatorname{div}(\mathbf{F}(x,y,z)) = 0$.
- *For* (b), *get* 2/7 *by obtaining* (3); *get* 2/7 *by obtaining* (4); *get* 2/7 *by obtaining* (5); *get* 1/7 *by obtaining* (6).
- *For* (c), *get* 1/5 *by explaining the setting in the red paragraph; get* 1/5 *by obtaining each of* (7)–(10).

5. (a) (8 pts) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+2)}$.

(b) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Please state the test(s) that you use.

(i) (6 pts) $\sum_{n=1}^{\infty} (-1)^n \cdot \left[\frac{1}{n} + \tan^{-1}\left(\frac{1}{n}\right)\right]$.

(ii) (6 pts) $\sum_{n=1}^{\infty} (-1)^n \cdot (2^{1/n^2} - 1)$.

Solution:

(a) (Method 1: Use ratio test)

Set $a_n = \frac{x^{2n}}{(2n+1)(2n+2)}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+2}}{(2n+3)(2n+4)}}{\frac{x^{2n}}{(2n+1)(2n+2)}} \right| = |x^2|. \text{ (2\%)}$$

By ratio test, we know that $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+3)}$ converges for $|x| < 1$ (1%) and diverges for $|x| > 1$ (1%).

For $x = \pm 1$, we consider the series $\sum_{n=0}^{\infty} \frac{(\pm 1)^{2n}}{(2n+1)(2n+3)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)}$. We compute that

$$\lim_{n\to\infty} \frac{\frac{1}{(2n+1)(2n+3)}}{\frac{1}{n^2}} = \frac{1}{4}. \text{ (1\%)}$$

Since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, by limit comparison test, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)}$ converges (2%). Therefore, the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+3)}$ is $[-1, 1]$ (1%).

(Method 2: Use root test)

Set $a_n = \frac{x^{2n}}{(2n+1)(2n+2)}$. Then, by

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(2n+1)(2n+3)}} = \lim_{n\to\infty} e^{-\frac{\ln(2n+1)(2n+3)}{n}} = \lim_{n\to\infty} e^{-\frac{8n+8}{(2n+1)(2n+3)}} = e^0 = 1, \text{ (1\%)}$$

we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{(2n+1)(2n+3)}} |x^2| = |x^2|. \text{ (1\%)}$$

By root test, we know that $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+3)}$ converges for $|x| < 1$ (1%) and diverges for $|x| > 1$ (1%).

For $x = \pm 1$, we consider the series $\sum_{n=0}^{\infty} \dfrac{(\pm 1)^{2n}}{(2n+1)(2n+3)} = \sum_{n=0}^{\infty} \dfrac{1}{(2n+1)(2n+3)}$. We compute that

$$\lim_{n\to\infty} \frac{\frac{1}{(2n+1)(2n+3)}}{\frac{1}{n^2}} = \frac{1}{4}. \ (1\%)$$

Since $\sum_{n=0}^{\infty} \dfrac{1}{n^2}$ converges, by limit comparison test, $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)(2n+3)}$ converges (2%). Therefore, the interval of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n+1)(2n+3)}$ is $[-1, 1]$ (1%).

(b)-(i) Set $a_n = \left|(-1)^n \left(\dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}\right)\right| = \dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}$ by $\dfrac{1}{n}$, $\tan^{-1}\left(\dfrac{1}{n}\right) > 0$ for $n \geq 1$. Since $\sum_{n=0}^{\infty} \dfrac{1}{n}$ diverges and $\dfrac{1}{n} + \tan^{-1}\dfrac{1}{n} > \dfrac{1}{n}$, by comparison test, we have $\sum_{n=0}^{\infty} \dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}$ diverges (2%). So $\sum_{n=1}^{\infty} (-1)^n \left(\dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}\right)$ is not absolutely convergent (1%). Set $b_n = \dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}$. Let $f(x) = x + \tan^{-1} x$. Then we have $f'(x) = 1 + \dfrac{1}{1+x^2} > 0$ for $0 \leq x \leq 1$. (Or Since $\tan^{-1} x$ is increasing for $x \geq 0$, we have $\tan^{-1}(1/n)$ is decreasing.) So b_n is decreasing and $\lim_{n\to\infty} \dfrac{1}{n} + \tan^{-1}\dfrac{1}{n} = 0$ (1%). By alternating series test, we obtain that $\sum_{n=1}^{\infty} (-1)^n \left(\dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}\right)$ is convergent (1%). Therefore, $\sum_{n=1}^{\infty} (-1)^n \left(\dfrac{1}{n} + \tan^{-1}\dfrac{1}{n}\right)$ is conditionally convergent (1%).

(b)-(ii) Set $a_n = \left|(-1)^n (2^{1/n^2} - 1)\right| = 2^{1/n^2} - 1$. Then we compute that

$$\lim_{n\to\infty} \frac{2^{1/n^2} - 1}{1/n^2} = \lim_{x\to 0^+} \frac{2^x - 1}{x} = \lim_{x\to 0^+} \ln 2 \cdot 2^x = \ln 2. (2\%)$$

Since $\sum_{n=1}^{\infty} (1/n^2)$ converges (1%), by limit comparison test (1%), we obtain that $\sum_{n=1}^{\infty} (2^{1/n^2} - 1)$ converges (1%). Therefore, $\sum_{n=1}^{\infty} (-1)^n (2^{1/n^2} - 1)$ is absolutely convergent (1%).

6. Consider the function $f(x) = \int_0^x e^{-t^3}\,dt$.

(a) (6 pts)Write down the Maclaurin series of $f(x)$ and specify its radius of convergence.

(b) (4 pts)What is the value of $f^{(691)}(0)$?

(c) (5 pts)Evaluate $\lim_{x\to 0} \frac{f(x) - x}{(e^{2x^2} - 1)\cdot \sin(3x^2)}$.

(d) (5 pts)Approximate the value of $f(0.5)$ up to an error of 10^{-4} by some estimation theorem of series.

Solution:

(a)

Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ (1 pt) for $t \in \mathbb{R}$, we have

$$e^{-t^3} = \sum_{n=0}^{\infty} \frac{(-t^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot t^{3n} \quad \text{for} \quad t \in \mathbb{R} \text{ (2 pts)}$$

and

$$f(x) = \int_0^x e^{-t^3} dt = \int_0^x (\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot t^{3n}) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \int_0^x t^{3n} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)\cdot n!} \cdot (t^{3n+1} \Big|_0^x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)\cdot n!} x^{3n+1} \text{ for } \ x \in \mathbb{R}. \text{ (2 pts)}$$

The radius of convergence of $f(x)$ is ∞. (1 pt)

(b)

By Taylor theorem, we need to solve the equation $3n+1 = 691 \Rightarrow n = 230$. (2 pts) The coefficient of x^{691} in the Maclaurin series of $f(x)$ is $\frac{(-1)^{230}}{691\cdot 230!}$. So we have

$$f^{(691)}(0) = \frac{(-1)^{230}}{691\cdot 230!} \cdot (691!) = \frac{690!}{230!}. \text{ (2 pts)}$$

(c)

(Method I)

Since

$$e^{2x^2} = \sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot x^{2n} = 1 + 2x^2 + 2x^4 + \cdots, \text{ (1 pt)}$$

$$\sin(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (3x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)!} \cdot x^{4n+2} = 3x^2 - \frac{9}{2}x^6 + \cdots, \text{ (1 pt)}$$

we have

$$\lim_{x\to 0} \frac{f(x) - x}{(e^{2x^2} - 1)\cdot \sin(3x^2)} = \lim_{x\to 0} \frac{(x - \frac{1}{4}x^4 + \cdots) - x}{[(1 + 2x^2 + 2x^4 + \cdots) - 1]\cdot (3x^2 - \frac{9}{2}x^6 + \cdots)} \text{ (1 pt)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{4}x^4 + x^5(\cdots)}{6x^4 + x^5(\cdots)} = \frac{-1}{24}. \text{ (2 pts)}$$

(Method II)

$$\lim_{x\to 0} \frac{f(x) - x}{(e^{2x^2} - 1)\cdot \sin(3x^2)} = \lim_{x\to 0} \frac{\int_0^x e^{-t^3} dt - x}{(e^{2x^2} - 1)\cdot \sin(3x^2)} \quad \text{(fumdamental theorem of calculus)}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{e^{-x^3} - 1}{e^{2x^2}\cdot (4x)\cdot \sin(3x^2) + (e^{2x^2} - 1)\cdot \cos(3x^2)\cdot 6x} \text{ (2 pts)}$$

Let $g(x) := e^{2x^2} \cdot (4x) \cdot \sin(3x^2)$, $h(x) := (e^{2x^2} - 1) \cdot \cos(3x^2) \cdot 6x$. Then we have

$$g'(x) = e^{2x^2} \cdot (4x)^2 \cdot \sin(3x^2) + e^{2x^2} \cdot 4 \cdot \sin(3x^2) + e^{2x^2} \cdot (4x) \cdot \cos(3x^2) \cdot (6x)$$
$$h'(x) = e^{2x^2} \cdot 4x \cdot \cos(3x^2) \cdot 6x - (e^{2x^2} - 1) \cdot \sin(3x^2) \cdot (6x)^2 + (e^{2x^2} - 1) \cdot \cos(3x^2) \cdot 6$$
$$\lim_{x \to 0} \frac{g'(x)}{x^2} = 0 + 12 + 24 = 36$$
$$\lim_{x \to 0} \frac{h'(x)}{x^2} = 24 + 0 + 12 = 36$$

Hence

$$\lim_{x \to 0} \frac{f(x) - x}{(e^{2x^2} - 1) \cdot \sin(3x^2)} = \lim_{x \to 0} \frac{e^{-x^3} - 1}{e^{2x^2} \cdot (4x) \cdot \sin(3x^2) + (e^{2x^2} - 1) \cdot \cos(3x^2) \cdot 6x}$$
$$\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{-3x^2 \cdot e^{-x^3}}{g'(x) + h'(x)} = \lim_{x \to 0} \frac{-3e^{-x^3}}{\frac{g'(x)}{x^2} + \frac{h'(x)}{x^2}} = \frac{-3}{36 + 36} = \frac{-1}{24}.$$ (3 pts)

(d)
$f(0.5) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1) \cdot n!} (0.5)^{3n+1}$ (1 pt) is an alternating series. Let R_N be the error incurred by estimating with the N-th partial sum. Then

$$R_N \leq \frac{1}{(3N+4) \cdot (N+1)!} \cdot (0.5)^{3N+4}.$$ (2 pts)

Let $\frac{1}{(3N+4) \cdot (N+1)!} \cdot (0.5)^{3N+4} \leq 10^{-4}$. Then we get $N \geq 2$. (1 pt) Therefore,

$$f(0.5) \approx \frac{1}{2} - \frac{1}{64} + \frac{1}{7 \cdot 256}.$$ (1 pt)

Note. If you only give(guess) a number without any explaination, then you get 0 credit.