1. (18 pts) Let $f(x,y) = \ln(y^2 + e^x) + 2y - 3$.
   (a) (4 pts) Compute $\nabla f(x,y)$.
   (b) (3 pts) Find the directional derivative of $f(x,y)$ at $(0,0)$ in the direction $\mathbf{u} = \langle 3,4\rangle$.
   (c) (3 pts) At the point $(0,0)$, in what direction does $f(x,y)$ have the maximum rate of change? Find the maximum rate of change.
   (d) (4 pts) Write down the linear approximation of $f(x,y)$ at $(0,0)$. Use the linear approximation to estimate $f(0.06, 0.08)$.
   (e) (4 pts) Let $F(x,y,z) = f(x,y) + (\tan z)^2$. Find the tangent plane of the level surface $F(x,y,z) = 0$ at $(0,0,\frac{\pi}{3})$.

Solution:

(a)
$$\nabla f = \langle \frac{e^x}{y^2+e^x}\ (2\text{ pts}), \frac{2y}{y^2+e^x} + 2\ (2\text{ pts})\rangle.$$

(b) The vector $\langle 3,4\rangle$ has length 5, hence the unit vector in this direction is $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$ (1 pt). The directional derivative is
$$f_x(0,0)\cdot\frac{3}{5} + f_y(0,0)\cdot\frac{4}{5} = \frac{11}{5}.\ (2\text{ pts})$$

(c) Since $\nabla f(0,0) = \langle 1,2\rangle$, the maximum change of rate of $f(x,y)$ at $(0,0)$ is $\sqrt{5}$ (2 pts), in the direction $\langle 1,2\rangle$. (1 pts)

(d)
$$L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = x + 2y - 3.\ (2\text{ pts})$$
One has $f(0.06, 0.08) \approx -2.78$. (2 pts)

(e)

Sol 1: We have
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}\ (1\text{ pt}), \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}\ (1\text{ pt})$$
So
$$\frac{\partial z}{\partial x}(0,0,\frac{\pi}{3}) = -\frac{1}{8\sqrt{3}}, \frac{\partial z}{\partial y}(0,0,\frac{\pi}{3}) = -\frac{2}{8\sqrt{3}} = -\frac{1}{4\sqrt{3}}.$$
Thus the tangent plane of $F(x,y,z) = 0$ at the point $(0,0,\frac{\pi}{3})$ is
$$z = \frac{\pi}{3} - \frac{1}{8\sqrt{3}}x - \frac{1}{4\sqrt{3}}y,\ \text{or}\ x + 2y + 8\sqrt{3}(z - \frac{\pi}{3}) = 0.\ (2\text{ pts})$$

Sol 2:
$$\nabla F(x,y,z) = \langle \frac{e^x}{y^2+e^x}, \frac{2y}{y^2+e^x} + 2, 2\tan z \sec^2 z\rangle\ (2\text{ pts}),$$
so $\nabla F(0,0,\frac{\pi}{3}) = \langle 1,2,8\sqrt{3}\rangle$. Since the tangent plane is perpendicular to ∇F, it is defined by $x + 2y + 8\sqrt{3}(z - \frac{\pi}{3}) = 0$ (2 pts).

2. (12 pts) Let $f(x,y) = x^4 - 2x^2 - 2xy^2 - y^2$.

   (a) (5 pts) Find all five critical points of $f(x,y)$.

   (b) (7 pts) Use the second derivatives test to classify all the critical points.

Solution:

(a) We have

$$f_x(x,y) = 4x^3 - 4x - 2y^2 = 4(x^3 - x) - 2y^2 \text{ (1 pt)}, \quad f_y = -4xy - 2y = -(4x+2)y \text{ (1 pt)}.$$

So $f_y(x,y) = 0$ if $y = 0$ or $x = -\frac{1}{2}$. When $x = -\frac{1}{2}$, one has $f_x(-\frac{1}{2}, y) = \frac{3}{2} - 2y^2$. Hence $f_x(-\frac{1}{2}, y) = 0$ if and only if $y = \pm\frac{\sqrt{3}}{2}$. When $y = 0$ then $f_x(x,0) = 4(x^3 - x) = 4x(x-1)(x+1)$, so $f_x(x,0) = 0$ if and only if $x = 0$ or ± 1.

In conclusion, there are five critical points $(0,0)$, $(1,0)$, $(-1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. (0.6 pt each, round off to integer)

(b) We have

$$f_{xx}(x,y) = 4(3x^2 - 1) \text{ (1 pt)}, \quad f_{xy}(x,y) = -4y \text{ (1 pt) and } f_{yy} = -(4x+2) \text{ (1 pt)}.$$

Hence

(x,y)	f_{xx}	f_{xy}	f_{yy}	D
$(0,0)$	-4	0	-2	8
$(1,0)$	8	0	-6	-48
$(-1,0)$	8	0	2	16
$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	-1	$-2\sqrt{3}$	0	-12
$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	-1	$2\sqrt{3}$	0	-12

Thus $(0,0)$ is a local maximum (1 pt), $(-1,0)$ is a local minimum (1 pt), and $(1,0)$, $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are saddle points (0.66 pt each, round off to integer).

3. (12 pts) An example of the Cobb-Douglas production function is given in the form

$$P(x, y, z) = x(y)^{1/3}(z)^{1/6}, \quad x\text{: technology, } y\text{: labor input, } z\text{: capital input.}$$

(a) (8 pts) For x, y, $z > 0$, use the method of Lagrange multiplies to find the possible local extreme value of $P(x, y, z)$ on the constraint set $3x + y + 2z = c$, where $c > 0$ is a constant. The local extreme value is the maximum value of $P(x, y, z)$ under the constraint $3x + y + 2z = c$, and we denote it by $P_{\max}(c)$.

(b) (4 pts) Compute $\frac{\mathrm{d}}{\mathrm{d}c}P_{\max}(c)$. Use the linear approximation at $c = 100$ to estimate $P_{\max}(101) - P_{\max}(100)$.

Solution:

(a) Let $g(x, y, z) = 3x + y + 2z - c$, then $\nabla P = \lambda \nabla g$ give us the equations

$$(y)^{1/3}(z)^{1/6} = 3\lambda, \quad \frac{1}{3}x(y)^{-2/3}(z)^{1/6} = \lambda, \quad \frac{1}{6}x(y)^{1/3}(z)^{-5/6} = 2\lambda$$

Since $x, y, z > 0$, we can obtain $xy^{1/3}z^{1/6} = 3x\lambda = 3y\lambda = 12z\lambda$. We can also see that λ cannot be zero, hence $x = y = 4z$. From $3x + y + 2z = c$, we get

$$x = \frac{2c}{9}, y = \frac{2c}{9}, z = \frac{c}{18}.$$

With this we can obtain $P_{\max}(c) = \dfrac{2^{7/6}c^{3/2}}{27}$. □

A different method:

From $3x + y + 2z = c$, we get $z = \frac{1}{2}(c - 3x - y)$.

$P(x, y) = \dfrac{1}{\sqrt[6]{2}}xy^{1/3}(c - 3x - y)^{1/6}$

Solve

$P_x = \dfrac{1}{\sqrt[6]{2}}y^{1/3}\left((c - 3x - y)^{1/6} - \dfrac{1}{2}x(c - 3x - y)^{-5/6}\right) = 0$

$P_y = \dfrac{1}{\sqrt[6]{2}}x\left(\dfrac{1}{3}y^{-2/3}(c - 3x - y)^{1/6} - \dfrac{1}{6}y^{1/3}(c - 3x - y)^{-5/6}\right) = 0$

Hence we can find the only critical point

$$2(c - 3x - y) = x, \quad 2(c - 3x - y) = y \quad \Rightarrow \quad x = y = \frac{2c}{9}$$

This method doesn't us the Lagrange multiplier method, so it won't get full credit. □

(b) $P'_{\max}(c) = \frac{2^{1/6}}{9}\sqrt{c}$

At $c = 100$, $P'_{\max}(100) = \frac{10}{9}\sqrt[6]{2}$.

Since $P_{\max}(101) - P_{\max}(100) \approx P'_{\max}(100)(101 - 100) = \frac{10}{9}\sqrt[6]{2}$. □

Grading:

2 points for knowing the method of Lagrange multiplier (Writing out all equations to solve).

2 points for the correct partial derivatives of P.

2 points for the correct partial derivatives of the constraint.

2 points for solving the system correctly and plugging the result into P.

2 points for finding the derivative of $P_{\max}(c)$.

2 points for knowing the derivative at $c = 100$ is the answer.

Notes:

• Partial credit should be considered for every step after the first. Part (b) depends on part (a)'s answer so they need to use whatever they write down.

• They do not need to write down the formula for the linear approximation. But if they write down the formula they can get 1 point.

• Only 1 point off if a clear copying or computation mistake occurs.

• Most students got (a) correct but was confused with (b). If they do not have a function of c, then they would lose at least 4 points.

4. (16 pts) Sketch the region of integration, change the order of integration, and evaluate it.

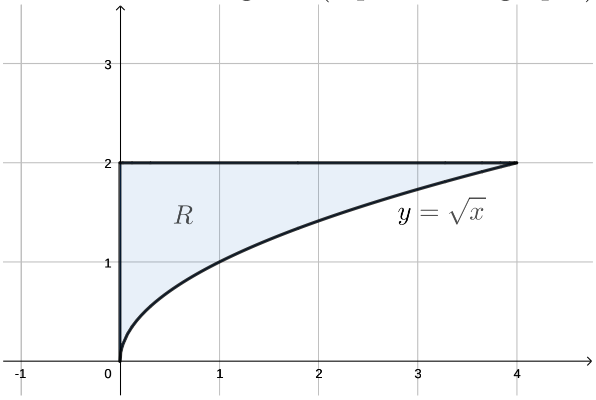
(a) (8 pts) $\int_0^4 \int_{\sqrt{x}}^2 \sin\left(\frac{x}{y}\right) \mathrm{d}y\, \mathrm{d}x$.

(b) (8 pts) $\int_1^2 \int_{\sqrt{y}}^y \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) \mathrm{d}x\, \mathrm{d}y + \int_2^4 \int_{\sqrt{y}}^2 \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) \mathrm{d}x\, \mathrm{d}y$.

Solution:

(a) $\int_0^4 \int_{\sqrt{x}}^2 \sin\left(\frac{x}{y}\right) dydx = \iint_R \sin\left(\frac{x}{y}\right) dA$

where R is the region: (1 pt for the graph.)



$$\iint_R \sin\left(\frac{x}{y}\right) dA = \int_0^2 \int_0^{y^2} \sin\left(\frac{x}{y}\right) dxdy \quad \text{(2 pts for correct upper limits)}$$

$$= \int_0^2 \left(-y\cos\left(\frac{x}{y}\right)\Big|_{x=0}^{x=y^2} \right) dy = \int_0^2 -y\cos y + y\, dy$$

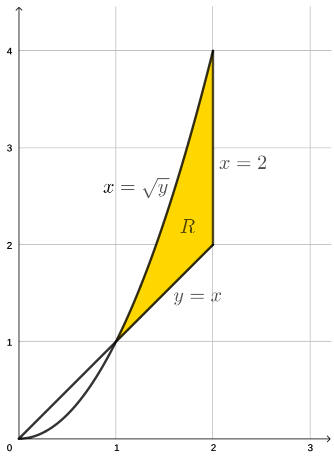
(2 pts for integration with respect to x.)

$$= -y\sin y\Big|_{y=0}^{y=2} + \int_0^2 \sin y\, dy + \frac{y^2}{2}\Big|_{y=0}^{y=2}$$

(2 pts for integration by parts.)

$$= 3 - 2\sin 2 - \cos 2 \quad \text{(1 pt for final answer)}$$

(b) $\int_1^2 \int_{\sqrt{y}}^y \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dxdy + \int_2^4 \int_{\sqrt{y}}^2 \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dxdy = \iint_R \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dA$, where R is



(2 pts for the graph)

$$\iint_R \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dA = \int_1^2 \int_x^{x^2} \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dydx \quad \text{(3 pts for correct ranges.)}$$

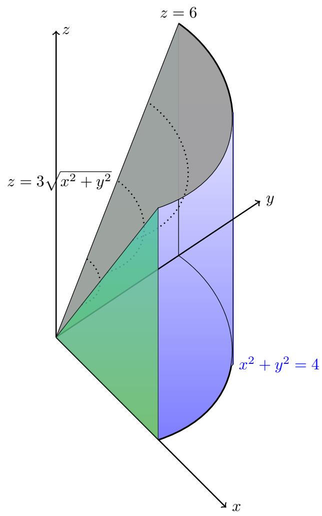
$$= \int_1^2 (x^2 - x)\cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dx$$ (1 pt for integration with respect to y)

$$\overset{\text{let } u = \frac{x^3}{3} - \frac{x^2}{2}}{\underset{du = (x^2 - x)dx}{=\!=\!=\!=}} \int_{-\frac{1}{6}}^{\frac{2}{3}} \cos(u)\, du$$

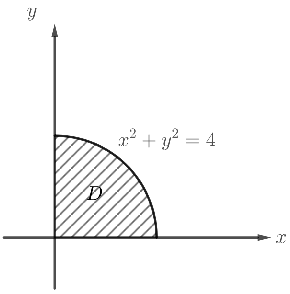
(1 pt for substitution and correct upper and lower bounds.)

$$= \sin\left(\frac{2}{3}\right) - \sin\left(-\frac{1}{6}\right) = \sin\left(\frac{2}{3}\right) + \sin\left(\frac{1}{6}\right)$$ (1 pt for final answer.)

5. (10 pts) Let E be the solid lying in the first octant within the cylinder $x^2 + y^2 = 4$ and below the cone $z = 3\sqrt{x^2 + y^2}$. Evaluate $\displaystyle\iiint_E x \, \mathrm{dV}$.



Solution:



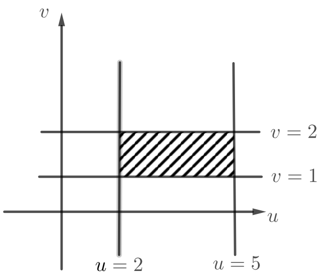
$$
\begin{aligned}
\iiint_E x\,dV &= \iint_D \left(\int_0^{3\sqrt{x^2+y^2}} x\,dz \right) dA \\
&= \iint_D 3\sqrt{x^2+y^2}\,x\,dA \quad \text{(3 points)} \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 3r(r\cos\theta) r\,dr d\theta \quad \text{(3 points)} \\
&= \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta \int_0^2 3r^3\,dr \\
&= 1 \cdot \frac{3}{4} r^4 \Big|_0^2 = 12 \quad \text{(4 points)}
\end{aligned}
$$

6. (10 pts) Evaluate the double integral $\iint_R \sqrt{\frac{y}{x}} e^{xy}\, dA$, where R is the region bounded by $xy = 2$, $xy = 5$, $y = x$, $y = 2x$ in the first quadrant.

Solution:

$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} = \frac{1}{2v}$$ (2 points for the formula of change of variables and Jacobian)



(2 points for the new region of integration)

$$\begin{aligned} \iint_R \sqrt{\frac{y}{x}} e^{xy}\, dA &= \int_2^5 \int_1^2 \sqrt{v} e^u \left(\frac{1}{2v}\right) dv du \quad \text{(3 points)} \\ &= \int_1^2 \frac{1}{2\sqrt{v}} dv \int_2^5 e^u\, du \\ &= \sqrt{v}\Big|_1^2 e^u\Big|_2^5 = (\sqrt{2} - 1)(e^5 - e^2) \quad \text{(3 points)} \end{aligned}$$

7. (10 pts) Assume that random variables X and Y are independent with probability density functions $f_X(x) = \begin{cases} e^{-x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$, $f_Y(x) = \begin{cases} e^{-y}, & \text{for } y \geq 0 \\ 0, & \text{for } y < 0 \end{cases}$. Let $Z = 2X + Y$.

For $z > 0$, $P(Z \leq z) = \iint_R f_X(x) \cdot f_Y(y)\, \mathrm{d}A$, where R is the region bounded by $2x + y = z$, $x = 0$ and $y = 0$.

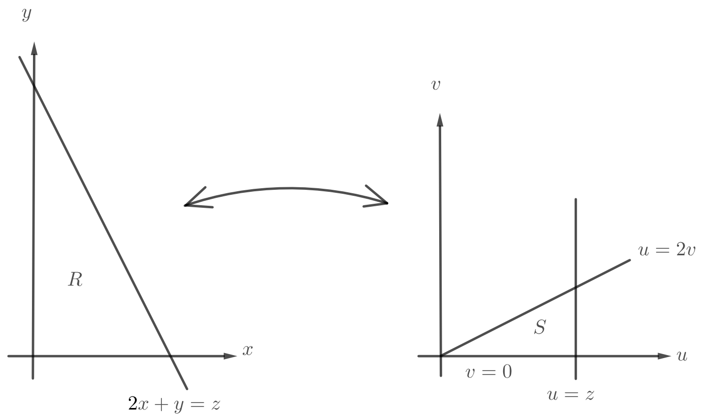
(a) (3 pts) For the change of variables $u = 2x+y$, $v = x$, sketch R in the xy-plane and its corresponding region S in the uv-plane.

(b) (2 pts) Find the Jacobian of the transformation from part (a).

(c) (5 pts) Write $P(Z \leq z)$ as $\iint\limits_S g(u,v)\mathrm{d}v\,\mathrm{d}u$. Then find $\frac{\mathrm{d}}{\mathrm{d}z}P(Z \leq z)$ which is the probability density function of Z.

Solution:

(a)



$$\begin{pmatrix} \text{1 pt for the graph of } R \\ \text{1 pt for the boundary } u = z \text{ of } S \\ \text{1 pt for the boundary } u = 2v \text{ of } S \end{pmatrix}$$

(b) $\begin{cases} x = v \\ y = u - 2v \end{cases}$, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ (1 pt) $= \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$ (1 pt)

(c)

$$\begin{aligned} P(Z \leq z) &= \iint\limits_R e^{-x-y} dA \\ &= \iint\limits_S e^{-v-u+2v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA \quad \text{(1 pt)} \\ &= \int_0^z \int_0^{\frac{u}{2}} e^{-u+v} dv du \quad \text{(2 pts)} \\ &= \int_0^z e^{-u}\left(e^{\frac{u}{2}} - 1\right) du = \int_0^z e^{-\frac{u}{2}} - e^{-u}\, du \quad \text{(1 pt)} \end{aligned}$$

Hence $\frac{d}{dz}P(Z \leq z) = e^{-\frac{z}{2}} - e^{-z}$ for $z > 0$. (1 pt)

8. (12 pts) Write down the first four nonzero terms of the Taylor series at $x = 0$ for (a) $\int_0^x e^{-t^2}\mathrm{d}t$, (b) $\sqrt[3]{1+x}$, (c) $\ln(1+x^2)$.

Solution:

(a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots$$

$$\int_0^x e^{-t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \qquad \square$$

(b) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{6}x^3 + \cdots$

$$\sqrt[3]{1+x} = (1+x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n = 1 + \frac{x}{3} - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots \qquad \square$$

(c) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \qquad \square$$

The direct derivatives method:

(a) $f(x) = \int_0^x e^{-t^2}\, dt, \quad f(0) = 0$

$f'(x) = e^{-x^2}, \quad f'(0) = 1$

$f''(x) = -2xe^{-x^2}, \quad f''(0) = 0$

$f^{(3)}(x) = (-2 + 4x^2)e^{-x^2}, \quad f^{(3)}(0) = -2$

$f^{(4)}(x) = (12x - 8x^3)e^{-x^2}, \quad f^{(4)}(0) = 0$

$f^{(5)}(x) = (12 - 48x^2 + 16x^4)e^{-x^2}, \quad f^{(5)}(0) = 12$

$f^{(6)}(x) = (-120x + 160x^3 - 32x^5)e^{-x^2}, \quad f^{(6)}(0) = 0$

$f^{(7)}(x) = (-120 + 720x^2 - 480x^4 + 64x^6)e^{-x^2}, \quad f^{(7)}(0) = -120$

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \qquad \square$$

(b) $g(x) = \sqrt[3]{1+x}, \quad g(0) = 1$

$g'(x) = \frac{1}{3}(1+x)^{-2/3}, \quad g'(0) = \frac{1}{3}$

$g''(x) = \frac{-2}{9}(1+x)^{-5/3}, \quad g''(0) = \frac{-2}{9}$

$g^{(3)}(x) = \frac{10}{27}(1+x)^{-8/3}, \quad g^{(3)}(0) = \frac{10}{27}$

$$g(x) = 1 + \frac{x}{3} - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots \qquad \square$$

(c) $h(x) = \ln(1+x^2), \quad h(0) = 0$

$h'(x) = \frac{2x}{1+x^2}, \quad h'(0) = 0$

$$h''(x) = \frac{2 - 2x^2}{(1 + x^2)^2}, \quad h''(0) = 2$$

$$h^{(3)}(x) = \frac{-12x + 4x^3}{(1 + x^2)^3}, \quad h^{(3)}(0) = 0$$

$$h^{(4)}(x) = \frac{-12 + 72x^2 - 12x^4}{(1 + x^2)^4}, \quad h^{(4)}(0) = -12$$

$$h^{(5)}(x) = \frac{240x - 480x^3 + 48x^5}{(1 + x^2)^5}, \quad h^{(5)}(0) = 0$$

$$h^{(6)}(x) = \frac{240 - 3600x^2 + 3600x^4 - 240x^6}{(1 + x^2)^6}, \quad h^{(6)}(0) = 240$$

$$h^{(7)}(x) = \frac{1440(-7x + 35x^3 - 21x^5 + x^7)}{(1 + x^2)^7}, \quad h^{(7)}(0) = 0$$

$$h^{(8)}(x) = \frac{10080(-1 + 28x^2 - 70x^4 + 28x^6 - x^8)}{(1 + x^2)^8}, \quad h^{(8)}(0) = -10080$$

$$h(x) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$ □

Grading:

4 points for each series.

+1 point for every correct step toward the answer. The steps can be (1) use a formula (2) substitute (3) derivative (4) integrate (5) plug in n values.

Notes: Take 1 point off per type of mistake. If they use derivatives to find the answer, then full credit as long as they don't make mistakes. If they didn't notice "nonzero" in the problem, then depending on whether they show understanding they can get 2 or 3 points.