1. The intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = x$ is a space curve known as the Viviani's curve. One way to express this parametric curve is

$$\mathbf{r}(t) = \langle \sin^2 t, \sin t \cos t, \cos t \rangle \text{ where } 0 \le t \le 2\pi.$$

   (a) (5 pts) Find the unit tangent vector $\mathbf{T}(t)$.

   (b) (3 pts) Find an equation of the normal plane for Viviani's curve at the point $\left(\frac{3}{4}, -\frac{\sqrt{3}}{4}, -\frac{1}{2}\right)$.

   (c) (4 pts) Find its curvature $\kappa(\pi)$ when $t = \pi$.

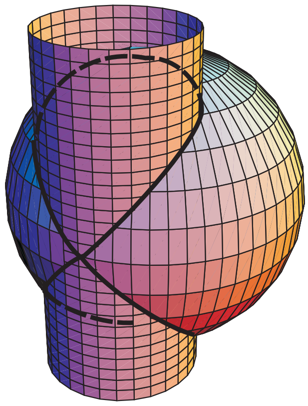

Figure : Viviani's curve.

Solution:

(a) $\mathbf{r}'(t) = \langle \sin 2t, \cos 2t, -\sin t \rangle$ or $\langle 2\sin t \cos t, \cos^2 t - \sin^2 t, -\sin t \rangle$ (1 pt)
$\|\mathbf{r}'(t)\| = \sqrt{1 + \sin^2 t}$ (1 pt)
$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+\sin^2 t}}\langle \sin 2t, \cos 2t, -\sin t \rangle$ or $\frac{1}{\sqrt{1+\sin^2 t}}\langle 2\sin t \cos t, \cos^2 t - \sin^2 t, -\sin t \rangle$ (3 pt: correct formula for $\mathbf{T}$ 2 pt, answer 1 pt))

Marking: (2nd line) if student use the second expression of $\mathbf{r}'(t)$ and simplify until $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^4 t + \sin^4 t + 2\sin^2 t \cos^2 t}$, still consider correct. (1 pt)

(b) Solve $(\sin^2 t, \sin t \cos t, \cos t) = \left(\frac{3}{4}, -\frac{\sqrt{3}}{4}, -\frac{1}{2}\right)$ and obtain $t = \frac{2\pi}{3}$. (1 pt)
Vector normal to the normal plane at $t = \frac{2\pi}{3}$ is $\mathbf{T}(\frac{2\pi}{3}) = \langle -\sqrt{\frac{3}{7}}, -\frac{1}{\sqrt{7}}, -\sqrt{\frac{3}{7}} \rangle$. (1 pt)
Equation of normal plane: $\sqrt{3}x + y + \sqrt{3}z = 0$ (1 pt)

Marking: (2nd line) any direction parallel to $\mathbf{T}(2\pi/3)$ is acceptable (1 pt).
Solve the wrong t but plug in the correct equation of normal plane (2 pt)

(c) $\mathbf{r}''(\pi) = \langle 2, 0, 1 \rangle$ (1 pt)
$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 0, 1, 0 \rangle \times \langle 2, 0, 1 \rangle = \langle 1, 0, -2 \rangle$ (1 pt),
$\|\mathbf{r}'(\pi)\| = 1$
$\kappa(\pi) = \frac{|r'(\pi) \times r''(\pi)|}{|r'(\pi)|^3} = \sqrt{5}$ (2 pt: Correct formula 1 pt, answer 1 pt)

2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2\sin(x)}{x^2-xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) (5 pts) Show that f is continuous at $(0,0)$.

(b) (5 pts) Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector. Find the directional derivative $D_{\mathbf{u}}f(0,0)$ in terms of a and b.

(c) (4 pts) By appealing to your answer in (b), prove that $f(x,y)$ is not differentiable at $(0,0)$.

Solution:

(a)

Marking scheme.

(1M) *Attempt to compute the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin(x)}{x^2-xy+y^2}$

(1M) Use of the fact $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$

(2M) **Correct method to evaluate $\displaystyle\lim_{x\to 0}\frac{x^3}{x^2-xy+y^2}$ (e.g. by Polar coordinates & Squeeze Theorem)

(1M) Correct limit by using a *correct* method

Remarks.

1. If a student attempts by taking paths, he/she can get at most 1M overall from *. (Note it's clear from the question that the relevant limit exists)

2. If a student writes $\displaystyle\lim_{x\to 0}\frac{x^3}{x^2-xy+y^2} = 0$ (or any equivalent limits) without any appropriate explanations, no marks will be awarded from **.

3. No deductions for students not explaining why $\dfrac{\cos^3\theta}{1-\cos\theta\sin\theta}$ is bounded. (But students are expected to at least mention it before applying the Squeeze Theorem)

4. L'Hospital's rule is not applicable. (note that θ can be implicitly dependent on r)

Sample Solution 1.

$$\underbrace{\lim_{(x,y)\to(0,0)}\frac{x^2\sin(x)}{x^2-xy+y^2}}_{\text{(1M)}} = \lim_{(x,y)\to(0,0)}\frac{x^3}{x^2-xy+y^2}\cdot\frac{\sin x}{x}$$

Passing to polar coordinates, we have

$$\lim_{(x,y)\to(0,0)}\frac{x^3}{x^2-xy+y^2} = \lim_{r\to 0^+} r\cdot\frac{\cos^3\theta}{1-\cos\theta\sin\theta} = \lim_{r\to 0^+} r\cdot\underbrace{\frac{\cos^3\theta}{1-\frac{1}{2}\sin(2\theta)}}_{\text{bounded}} = 0 \text{ by Squeeze Theorem}$$

(2M)

Hence, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2\sin(x)}{x^2-xy+y^2} = \lim_{(x,y)\to(0,0)}\frac{x^3}{x^2-xy+y^2}\cdot\frac{\sin x}{x} = 0\cdot\underbrace{1}_{\text{(1M)}} = \underbrace{0}_{\text{(1M)}} = f(0,0)$

Therefore, $f(x,y)$ is continuous at $(0,0)$.

Sample Solution 2.

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2\sin(x)}{x^2-xy+y^2}}_{(1M)} = \lim_{r\to 0^+} \left(\underbrace{r}_{\to 0} \cdot \underbrace{\frac{\cos^3\theta}{1-\frac{1}{2}\sin(2\theta)}}_{\text{bounded}} \right) \cdot \frac{\sin(r\cos\theta)}{r\cos\theta} = \underbrace{0}_{\text{Squeeze}(2M)} \cdot \underbrace{1}_{(1M)} = \underbrace{0}_{(1M)} = f(0,0).$$

Therefore, $f(x,y)$ is continuous at $(0,0)$.

Sample Solution 3.

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2\sin(x)}{x^2-xy+y^2}}_{(1M)} = \lim_{r\to 0^+} \frac{\cos^2\theta}{1-\frac{1}{2}\sin(2\theta)} \cdot \sin(r\cos\theta)$$

Since $\lim_{r\to 0^+} \sin(r\cos\theta) = \sin(\lim_{r\to 0^+} r\cos\theta) = 0$ (1M) and $\frac{\cos^2\theta}{1-\frac{1}{2}\sin(2\theta)}$ is a bounded function, we have by Squeeze Theorem that

$$\lim_{r\to 0^+} \frac{\cos^2\theta}{1-\frac{1}{2}\sin(2\theta)} \cdot \sin(r\cos\theta) = 0 = f(0,0). \quad \text{(2+1M, Correct argument)}$$

Therefore, $f(x,y)$ is continuous at $(0,0)$.

(b)

Marking scheme.
(1M) Correct definition of $D_{\mathbf{u}}f(0,0)$
(2M) Correct simplification of the expression $D_{\mathbf{u}}f(0,0)$
(2M) Correct evaluation of $D_{\mathbf{u}}f(0,0)$

Remarks.
1. No marks are awarded to students who use (incorrectly) that $D_{\mathbf{u}}f(0,0) = \nabla f(0,0)\cdot\mathbf{u}$.
2. No deductions for students not separating the case $a=0$ and leave $\frac{a^3}{1-ab}$ as the final answer.
3. No deductions for students not simplifying $a^2+b^2=1$.

Sample Solution.
By definition,

$$\begin{aligned} D_{\mathbf{u}}f(0,0) &= \lim_{t\to 0} \frac{f(ta,tb)-f(0,0)}{t}, \text{ (1M)} \\ &= \lim_{t\to 0} \frac{\frac{t^2a^2\sin(ta)}{t^2(a^2-ab+b^2)}-0}{t} = \lim_{t\to 0} \frac{a^2\sin(ta)}{t(1-ab)}, \text{ (2M)} \\ &= \begin{cases} 0 & \text{if } a=0 \\ \lim_{t\to 0} \frac{a^3}{1-ab}\cdot\frac{\sin(ta)}{ta} = \frac{a^3}{1-ab} & \text{if } a\neq 0 \end{cases} \quad \text{(2M)} \end{aligned}$$

(c)

Marking scheme.
(1M+1M) **Correct computation of $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0)\rangle$
(1M) Pick an explicit $\mathbf{u} = \langle a, b\rangle$ and find $\nabla f(0,0)\cdot\mathbf{u}$
(1M) Establishing that $D_{\mathbf{u}}f(0,0) \neq \nabla f(0,0)\cdot\mathbf{u}$ for some explicit $\mathbf{u}$

Remarks.
1. 3M for students who, without offering a concrete $\mathbf{u}$, write '$\frac{a^3}{1-ab}$ is not of the form $ac_1 + bc_2$ for some constants c_1, c_2' or '$\frac{a^3}{1-ab}$ is not a (bi)linear function in a and b'.
2. Marks for ** can be awarded even if a student wrote them in his/her work in (a) or (b)
3. The last 2M will not be awarded if (b) is incorrect.
4. We accept students for writing 'a is not identically equal to $\frac{a^3}{1-ab}$'.
5. We also accept proofs of non-differentiability by definition. In this case, the last 1M+1M will be allocated to a correct proof that $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}$ does not exist/does not vanish.

Sample Solution 1. (By using (b))
By (b), we have that $\underbrace{f_x(0,0)=1}_{\text{(1M)}}$ and $\underbrace{f_y(0,0)=0}_{\text{(1M)}}$.

Then we take, for example, $\mathbf{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$. We have that

$$\nabla f(0,0)\cdot\mathbf{u} = \frac{1}{2}\cdot 1 + \frac{\sqrt{3}}{2}\cdot 0 = \frac{1}{2} \quad \text{(1M)}$$

but

$$D_{\mathbf{u}}f(0,0) = \frac{(1/2)^3}{1-(1/2)(\sqrt{3}/2)} = \frac{1}{8-2\sqrt{3}}$$

Therefore, $D_{\mathbf{u}}f(0,0) \neq \nabla f(0,0)\cdot\mathbf{u}$ (1M) and hence the function is not differentiable at $(0,0)$.

Sample Solution 2. (By definition of differentiability)
By (b), we have that $\underbrace{f_x(0,0)=1}_{\text{(1M)}}$ and $\underbrace{f_y(0,0)=0}_{\text{(1M)}}$.

Therefore, the linearisation of f at $(0,0)$ is given by $L(x,y)=x$. Now consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^2\sin(x)}{x^2-xy+y^2} - x}{\sqrt{x^2+y^2}}$$

Along the path $(0,t)$, we have

$$\frac{\frac{x^2\sin(x)}{x^2-xy+y^2} - x}{\sqrt{x^2+y^2}} = 0 \to 0 \text{ , as } t\to 0$$

Along the path $(t,2t)$ with $t>0$, we have

$$\frac{\frac{x^2\sin(x)}{x^2-xy+y^2} - x}{\sqrt{x^2+y^2}} = \frac{\frac{1}{3}\sin t - t}{\sqrt{5}t} \to \frac{1}{3\sqrt{5}} - \frac{1}{\sqrt{5}} \text{ , as } t\to 0$$

Since the limit along two paths do not agree (2M), we conclude that $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}}$ does not exist and hence the function is not differentiable at $(0, 0)$.

3. Consider the level surface defined by the equation $z^5 + (\sin x)z^3 + yz = 4$.

(a) (5 pts) Find an equation of the tangent plane at $(0, 3, 1)$.

(b) The equation of the level surface defines $z = z(x, y)$ implicitly near $(0, 3, 1)$ as a differentiable function in x and y.

i. (4 pts) Find $\left.\dfrac{\partial z}{\partial x}\right|_{(x,y)=(0,3)}$ and $\left.\dfrac{\partial z}{\partial y}\right|_{(x,y)=(0,3)}$.

ii. (4 pts) Use the linear approximation of $z(x, y)$ at $(0, 3)$ to approximate the real root of the quintic polynomial $P(z) = z^5 + 2.9z - 4$.

Solution:

(a)

Marking scheme.
(2M) Find $\nabla F(x, y, z)$
(2M) Find $\nabla F(0, 3, 1)$
(1M) Equation of tangent plane

Remarks.
1. At most 4M can be awarded to students who made calculation error to at most one of the coordinates of ∇F.

Sample solution.
Let $F(x, y, z) = z^5 + (\sin x)z^3 + yz$. Then

$$\nabla F = \langle (\cos x)z^3,\ z,\ 5z^4 + 3(\sin x)z^2 + y \rangle. \quad \text{(2M)}$$

Therefore,

$$\nabla F(0, 3, 1) = \langle 1, 1, 8 \rangle. \quad \text{(2M)}$$

Hence,

$$\langle 1, 1, 8 \rangle \cdot \langle x, y - 3, z - 1 \rangle = 0 \quad \text{(1M)}$$

or $x + y + 8z = 11$ gives the equation of the tangent plane.

(b) (i)

Marking scheme.
(2M) Use of Implicit Function Theorem or Chain rule (correctly)
(1M) Correct value for $z_x(0, 3)$
(1M) Correct value for $z_y(0, 3)$

Remarks.
1. At most 1M will be taken away if a student has forgotten to evaluate these partial derivatives at $(0, 3)$

Sample solution.
By implicit function theorem,

$$\underbrace{z_x(0,3) = -\frac{F_x(0,3,1)}{F_z(0,3,1)}}_{\text{(2M)}} = \underbrace{-\frac{1}{8}}_{\text{(1M)}} \text{ and } z_y(0,3) = -\frac{F_y(0,3,1)}{F_z(0,3,1)} = \underbrace{-\frac{1}{8}}_{\text{(1M)}}.$$

(ii)

Marking scheme.
(2M) Writing down the linear approximation of $z(x, y)$
(1M+1M) Association with the substitution $(x, y) = (0, 2.9)$ and correct answer

Remarks.
1. 1M can be awarded to students who demonstrate some understandings of the concept of 'linear approximation in multivariables'.
2. At most 3M can be given to students who inherited an incorrect answer from (b)

Sample solution.
The required linear approximation is

$$L(x, y) = 1 - \frac{1}{8}x - \frac{1}{8}(y - 3). \quad \text{(2M)}$$

The root of the quintic polynomial $P(X) = X^5 + 2.9X - 4$ equals to $z(0, 2.9)$ (1M) which can be approximated by $L(0, 2.9) = \frac{81}{80}$ (1M).

4. Let $f(x,y) = \ln(x+y+1) + x^2 - y$.

(a) (7 pts) Find and classify the critical point(s) of f.

(b) (6 pts) Find the absolute maxima and minima of f on $D = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le 1\}$.

Solution:

(a) $f_x = \dfrac{1}{x+y+1} + 2x,\ f_y = \dfrac{1}{x+y+1} - 1.$ (1%)

To solve $f_x = 0$ and $f_y = 0$, we have $x = -1/2$ and $y = 1/2$. So the critical point of f is $(-1/2, 1/2)$. (2%)

$f_{xx} = \dfrac{-1}{(x+y+1)^2} + 2,\ f_{xy} = \dfrac{-1}{(x+y+1)^2}$ and $f_{yy} = \dfrac{-1}{(x+y+1)^2}$. (1%)

Then

$$D(-\frac{1}{2},\frac{1}{2}) = f_{xx}(-\frac{1}{2},\frac{1}{2})f_{yy}(-\frac{1}{2},\frac{1}{2}) - [f_{xy}(-\frac{1}{2},\frac{1}{2})]^2 = -2 < 0.(1\%)$$

So the critical point $(-1/2, 1/2)$ is a saddle point.(2%)

(b) Since $(-1/2, 1/2)$ is not in D, the extreme value of f does not occur in the interior of D. (1%)

The boundary of D consists of the four line segments $L_1 : x = 0,\ 0 \le y \le 1,\ L_2 : y = 0,\ 0 \le x \le 1,\ L_3 : x = 1,\ 0 \le y \le 1$ and $L_4 : y = 1,\ 0 \le x \le 1$. On L_1, we have

$$f_1(y) := f(0,y) = \ln(y+1) - y, \quad 0 \le y \le 1.$$

Since $f_1'(y) = \dfrac{-y}{y+1} \le 0$ for $0 \le y \le 1$, we have f_1 is decreasing in y. So its maximum value is $f(0,0) = 0$ and its minimum value is $f(0,1) = \ln 2 - 1$. (1%)

On L_2, we have

$$f_2(x) := f(x,0) = \ln(x+1) + x^2, \quad 0 \le x \le 1.$$

Since $f_2'(x) = \dfrac{1}{x+1} + 2x \ge 0$ for $0 \le x \le 1$, we have f_2 is increasing in x. So its maximum value is $f(1,0) = \ln 2 + 1$ and its minimum value is $f(0,0) = 0$. (1%)

On L_3, we have

$$f_3(y) := f(1,y) = \ln(y+2) + 1 - y, \quad 0 \le y \le 1.$$

Since $f_3'(y) = \dfrac{-y-1}{y+2} \le 0$ for $0 \le y \le 1$, we have f_3 is decreasing in y. So its maximum value is $f(1,0) = \ln 2 + 1$ and its minimum value is $f(1,1) = \ln 3$. (1%)

On L_4, we have

$$f_4(x) := f(x,1) = \ln(x+2) + x^2 - 1, \quad 0 \le x \le 1.$$

Since $f_4'(x) = \dfrac{1}{x+2} + 2x \ge 0$ for $0 \le x \le 1$, we have f_4 is increasing in x. So its maximum value is $f(1,1) = \ln 3$ and its minimum value is $f(0,1) = \ln 2 - 1$. (1%)

Therefore, the absolute maximum value of f on D is $\ln 2 + 1$ and the absolute minimum value of f on D is $\ln 2 - 1$. (1%)

5. (10 pts) Use the method of Lagrange multipliers to find the absolute maximum and minimum values (if exists) of $f(x,y,z)=x^2+3z^2$ subject to the constraints $x+y+z=1$ and $x-y+2z=2$.

Solution:

Let $g_1(x,y,z)=x+y+z$ and $g_2(x,y,z)=x-y+2z$. Then we will find

$$\text{Max } f(x,y,z), \quad \text{Min } f(x,y,z)$$

subject to the constraints

$$\begin{cases} g_1(x,y,z) & = x+y+z=1, \\ g_2(x,y,z) & = x-y+2z=2. \end{cases} \qquad \text{(2 pts)}$$

By Lagrange multipliers, we have

$$\begin{cases} \nabla f & = \lambda \cdot \nabla g_1 + \mu \cdot \nabla g_2 \\ g_1(x,y,z) & = x+y+z=1 \\ g_2(x,y,z) & = x-y+2z=2 \end{cases}$$

$$\Rightarrow \begin{cases} (2x,0,6z) & = \lambda\cdot(1,1,1)+\mu\cdot(1,-1,2) \\ x+y+z & =1 \\ x-y+2z & =2 \end{cases} \Rightarrow \begin{cases} 2x & = \lambda+\mu \\ 0 & = \lambda-\mu \\ 6z & = \lambda+2\mu \\ x+y+z & =1 \\ x-y+2z & =2 \end{cases}$$

$$\Rightarrow \begin{cases} 2x & = \lambda+\mu \\ \lambda & = \mu \\ 6z & = \lambda+2\mu \\ x+y+z & =1 \\ x-y+2z & =2 \end{cases} \Rightarrow \begin{cases} x=\lambda=\mu & =2z \\ x+y+z & =1 \\ x-y+2z & =2 \end{cases} \Rightarrow \begin{cases} x & = \lambda=\mu=\frac{6}{7} \\ y & = \frac{-2}{7} \\ z & = \frac{3}{7}. \end{cases} \qquad \text{(5 pts)}$$

Note that the intersection of two planes $g_1(x,y,z)=1$ and $g_2(x,y,z)=2$ is a line L in $\mathbb{R}^3$. By Lagrange multiplier, the absolute minimum value of the function f is $f(\frac{6}{7},\frac{-2}{7},\frac{3}{7})=\frac{9}{7}$ since there is another point $(3,-1,-1)\in L$ with $f(3,-1,-1)=12>\Gamma f(\frac{6}{7},\frac{-2}{7},\frac{3}{7})$. (2 pts) There is no absolute maximum value since the x-coordinate (or z-coordinate) of the point on L can be arbitrary large. (1 pt)

Remark 1. (If you use the following method, you can only get at most 6 credits.)

Since

$$\begin{cases} x+y+z & =1 \\ x-y+2z & =2 \end{cases} \Rightarrow 2x+3z=3 \Rightarrow x=\frac{1}{2}(3-3z),$$

the function $f(x,y,z)$ becomes

$$f(x,y,z)=x^2+3z^2=(\frac{3}{2}-\frac{3}{2}z)^2+3z^2=\frac{21}{4}(z-\frac{3}{7})^2+\frac{9}{7}\geq\frac{9}{7}. \qquad \text{(2 pts)}$$

The absolute minimum value of the function f is $f(\frac{6}{7},\frac{-2}{7},\frac{3}{7})=\frac{9}{7}$. (3 pts) There is no absolute maximum value since the x-coordinate (or z-coordinate) of the point on L can be arbitrary large. (1 pt)

Remark 2. (If you use the following method, you can only get at most 6 credits.)

The intersection of two planes $x+y+z=1$ and $x-y+2z=2$ is a line L in $\mathbb{R}^3$ described by $(x,y,z)=(3+3t,-1-t,-1-2t)$ for each point $(x,y,z)\in L$. Then the function $f(x,y,z)$ becomes

$$f(x,y,z)=(3+3t)^2+3(-1-2t)^2=21(t+\frac{5}{7})^2+\frac{9}{7}\geq\frac{9}{7}. \qquad \text{(2 pts)}$$

Under the constraints, the function $f(x,y,z)$ obtains its absolute minimum value $\frac{9}{7}$ when $t = \frac{-5}{7}$. which implies $(x,y,z) = (\frac{6}{7}, \frac{-2}{7}, \frac{3}{7})$. (3 pts) There is no absolute maximum value since the x-coordinate (or z-coordinate) of the point on L can be arbitrary large. (1 pt)

Remark 3. (If you use the following method, you can only get at most 9 credits.)
Since

$$\begin{cases} x+y+z &= 1 \\ x-y+2z &= 2 \end{cases} \Leftrightarrow \begin{cases} 2x+3z &= 3 \\ x+y+z &= 1 \end{cases} \quad \text{(1 pt)}$$

we will use Lagrange multiplier method to find the extreme values of $f(x,y,z) := x^2 + 3z^2$ with the constraint $g(x,y,z) := 2x + 3z = 3$. (2 pts) By Lagrange multipliers, we have

$$\begin{cases} \nabla f &= \lambda \cdot \nabla g \\ g(x,y,z) &= 2x+3z = 3 \end{cases}$$

$$\Rightarrow \begin{cases} (2x,0,6z) &= \lambda \cdot (2,0,3) \\ 2x+3z &= 3 \end{cases} \Rightarrow \begin{cases} 2x &= 2\lambda \\ 6z &= 3\lambda \\ 2x+3z &= 3 \end{cases}$$

$$\Rightarrow \begin{cases} x = \lambda &= 2z \\ 2x+3z &= 3 \end{cases}$$

$$\Rightarrow (x,z,\lambda) = (\frac{6}{7}, \frac{3}{7}, \frac{6}{7}) \Rightarrow (x,y,z) = (\frac{6}{7}, \frac{-2}{7}, \frac{3}{7}). \quad \text{(2 pts)}$$

Under the constraints $\begin{cases} 2x+3z &= 1 \\ x+y+z &= 1, \end{cases}$ we get

$$f(x,y,z) = f(\frac{1}{2}(1-3z), 1-x-z, z) = (\frac{3}{2} - \frac{3}{2}z)^2 + 3z^2 = \frac{21}{4}(z - \frac{3}{7})^2 + \frac{9}{7} \geq \frac{9}{7} = f(\frac{6}{7}, \frac{-2}{7}, \frac{3}{7}).$$

The absolute minimum value of the function $f(x,y,z) = x^2 + 3z^2$ is $f(\frac{6}{7}, \frac{-2}{7}, \frac{3}{7}) = \frac{9}{7}$. (3 pts)

There is no absolute maximum value since the x-coordinate (or z-coordinate) the constraint $2x + 3z = 3$ can be arbitrary large. (1 pt)

6. Evaluate the following integrals.

(a) (8 pts)

$$\int_1^2 \int_{\sqrt{x}}^x \sin\left(\frac{\pi x}{2y}\right) dydx + \int_2^4 \int_{\sqrt{x}}^2 \sin\left(\frac{\pi x}{2y}\right) dydx.$$

(b) (8 pts)

$$\iiint_E \sqrt{z}\ \mathrm{d}V,$$

where E is the solid lying below the cone $z^2 = 4x^2 + 4y^2$, within the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

Solution:

(a) (Method I)

$$\begin{aligned}
&\int_1^2 \int_{\sqrt{x}}^x \sin\frac{\pi x}{2y} dydx + \int_2^4 \int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y} dydx \\
&= \int_1^2 \int_y^{y^2} \sin\frac{\pi x}{2y} dxdy \quad \text{(4 pts)} \\
&= \int_1^2 [\frac{-2y}{\pi} \cdot \cos\frac{\pi x}{2y} \Big|_{x=y}^{x=y^2}]dy \\
&= \int_1^2 [\frac{-2y}{\pi} \cdot (\cos\frac{\pi y}{2} - \cos\frac{\pi}{2})]dy \quad \text{(1 pt)} \\
&= \frac{-2}{\pi} \int_1^2 y \cdot \cos\frac{\pi y}{2} dy = \frac{-4}{\pi^2} \int_1^2 y\ d\sin\frac{\pi y}{2} \\
&= \frac{-4}{\pi^2}[y \cdot \sin\frac{\pi y}{2} \Big|_1^2 - \int_1^2 \sin\frac{\pi y}{2} dy] \quad \text{(2 pts)} \\
&= \frac{-4}{\pi^2}[-1 + \frac{2}{\pi}\cos\frac{\pi y}{2} \Big|_1^2] = \frac{4}{\pi^3}(2+\pi). \quad \text{(1 pt)}
\end{aligned}$$

(Method II)

$$\begin{aligned}
&\int_1^2 \int_{\sqrt{x}}^x \sin\frac{\pi x}{2y} dydx + \int_2^4 \int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y} dydx \\
&= \int_1^2 \int_y^2 \sin\frac{\pi x}{2y} dxdy + \int_1^2 \int_2^{y^2} \sin\frac{\pi x}{2y} dxdy \quad \text{(4 pts)} \\
&= \int_1^2 [\frac{-2y}{\pi} \cdot \cos\frac{\pi x}{2y} \Big|_{x=y}^{x=2}]dy + \int_1^2 [\frac{-2y}{\pi} \cdot \cos\frac{\pi x}{2y} \Big|_{x=2}^{x=y^2}]dy \\
&= \int_1^2 [\frac{-2y}{\pi} \cdot (\cos\frac{2\pi}{2y} - \cos\frac{\pi}{2})]dy + \int_1^2 [\frac{-2y}{\pi} \cdot (\cos\frac{\pi y}{2} - \cos\frac{2\pi}{2y})]dy \quad \text{(1 pt)} \\
&= \int_1^2 [\frac{-2y}{\pi} \cdot (\cos\frac{\pi y}{2} - \cos\frac{\pi}{2})]dy \\
&= \frac{-2}{\pi} \int_1^2 y \cdot \cos\frac{\pi y}{2} dy = \frac{-4}{\pi^2} \int_1^2 y\ d\sin\frac{\pi y}{2} \\
&= \frac{-4}{\pi^2}[y \cdot \sin\frac{\pi y}{2} \Big|_1^2 - \int_1^2 \sin\frac{\pi y}{2} dy] \quad \text{(2 pts)} \\
&= \frac{-4}{\pi^2}[-1 + \frac{2}{\pi}\cos\frac{\pi y}{2} \Big|_1^2] = \frac{4}{\pi^3}(2+\pi). \quad \text{(1 pt)}
\end{aligned}$$

(b) (Method I) The solid E can be described as

$$\begin{aligned} E &= \{(x,y,z) \mid \ x^2+y^2 \le 1,\ 0 \le z \le \sqrt{4(x^2+y^2)},\} \\ &= \{(r,\theta,z) \mid \ 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 2r\}. \quad \text{(3 pts)} \end{aligned}$$

Using cylinder coordinate, we have

$$\begin{aligned} &\iiint_E \sqrt{z}\ dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{2r} \sqrt{z} \cdot r\ dz\ dr\ d\theta \quad \text{(2 pts)} \\ &= \int_0^{2\pi} \int_0^1 [\ \frac{2}{3} z^{\frac{3}{2}}\ ] \Big|_{z=0}^{z=2r} \cdot r\ dr\ d\theta \\ &= \frac{2}{3} \int_0^{2\pi} \int_0^1 2\sqrt{2} \cdot r^{\frac{5}{2}} dr\ d\theta \quad \text{(2 pts)} \\ &= \frac{4\sqrt{2}}{3} \int_0^{2\pi} \frac{2}{7} r^{\frac{7}{2}} \Big|_{r=0}^{r=1} d\theta \\ &= \frac{8\sqrt{2}}{21} \cdot \int_0^{2\pi} 1\ d\theta = \frac{8\sqrt{2}}{21} \cdot 2\pi = \frac{16\sqrt{2}}{21}\pi. \quad \text{(1 pt)} \end{aligned}$$

(Method II)

$$\begin{aligned} &\iiint_E \sqrt{z}\ dV \\ &= \iint_{0 \le x^2+y^2 \le 1} \int_0^{\sqrt{4(x^2+y^2)}} \sqrt{z}\ dz\ dA \quad \text{(3 pts)} \\ &= \iint_{0 \le x^2+y^2 \le 1} \frac{2}{3} z^{\frac{3}{2}} \Big|_0^{\sqrt{4(x^2+y^2)}} dA \\ &= \frac{2}{3} \iint_{0 \le x^2+y^2 \le 1} 2\sqrt{2} \cdot (x^2+y^2)^{\frac{3}{4}}\ dA \quad \text{(2 pts)} \\ &\text{(let } x = r\cos\theta, y = r\sin\theta) \\ &= \frac{4\sqrt{2}}{3} \int_0^{2\pi} \int_0^1 r^{\frac{3}{2}} \cdot r\ dr\ d\theta \quad \text{(2 pts)} \\ &= \frac{4\sqrt{2}}{3} \cdot (\int_0^1 r^{\frac{5}{2}} dr) \cdot (\int_0^{2\pi} 1 d\theta) \\ &= \frac{4\sqrt{2}}{3} \cdot \frac{2}{7} \cdot 2\pi = \frac{16\sqrt{2}\pi}{21}. \quad \text{(1 pt)} \end{aligned}$$

7. Let E be an upper-half ball which occupies the region

$$E = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \le 16 \text{ and } z \ge 0\}$$

and whose density function is $\rho(x, y, z) = \sqrt{x^2 + y^2}$ for each point $(x, y, z) \in E$.

(a) (7 pts) Find the mass of E.

(b) (5 pts) Let $(0, 0, \bar{z})$ be the center of mass of E. Find the value of $\bar{z}$.

Solution:

(a) the formula of mass is

$$M = \iiint_E \rho(x, y, z)\,\mathrm{d}V. \tag{1}$$

Applying spherical coordinates yields

$$\begin{aligned} M &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 r\sin(\varphi) r^2 \sin(\varphi)\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \qquad (2) \\ &= 32\pi^2. \qquad (3) \end{aligned}$$

(b) the formula of $\overline{z}$ is

$$\overline{z} = \frac{\iiint_E z\rho(x, y, z)\,\mathrm{d}V}{M}. \tag{4}$$

Applying spherical coordinates yields

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 r\cos(\varphi) r\sin(\varphi) r^2 \sin(\varphi)\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \qquad (5) \\ &= \frac{2^{11}\pi}{15}. \qquad (6) \end{aligned}$$

Therefore,

$$\overline{z} = \frac{64}{15\pi}. \tag{7}$$

Grading Suggestion.

- *For* (a), *get* 1/7 *by obtaining* (1), 3/7 *by obtaining* (2), *and* 3/7 *by obtaining* (3).
- *For* (b), *get* 1/5 *by by obtaining* (4), 2/5 *by obtaining* (5), 1/5 *by obtaining* (6), *and* 1/5 *by obtaining* (7).

8. (10 pts) Find the volume of the solid lying below the plane $z = 9$ and above the surface $z = (3x + y - 1)^2 + (x - y)^2$.

Solution:

Let $D := \{(x, y) \mid (3x + y - 1)^2 + (x - y)^2 \le 9\}$ be the region enclosed by $(3x+y-1)^2+(x-y)^2 = 9$ in the xy-plane. Then the solid in the question can be described as

$$E := \{(x, y, z) \mid (x, y) \in D, \ (3x + y - 1)^2 + (x - y)^2 \le z \le 9\}.$$

So we have

$$\text{Volume of the solid} = \iiint_E 1 \, dV$$

$$= \iint_D \int_{(3x+y-1)^2+(x-y)^2}^{9} 1 \, dz \, dA \quad \text{(4 pts)}$$

$$= \iint_D [9 - (3x + y - 1)^2 - (x - y)^2] \, dA \quad \text{(1 pt)}$$

$$\left(\text{let} \begin{cases} u = 3x + y - 1 \\ v = x - y \end{cases} . \Rightarrow \begin{cases} x = \frac{1}{4}(u + v + 1) \\ y = \frac{1}{4}(u - 3v + 1) \end{cases} \Rightarrow |J| = \left| \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-3}{4} \end{vmatrix} \right| = \frac{1}{4}. \right)$$

$$= \iint_{u^2+v^2\le 9} [9 - u^2 - v^2] \cdot \frac{1}{4} \, dA \quad \text{(3 pts)}$$

(let $u = r\cos\theta, v = r\sin\theta$)

$$= \frac{1}{4} \int_0^{2\pi} \int_0^3 [9 - r^2] \cdot r \, dr \, d\theta$$

$$= \frac{1}{4} \cdot 2\pi \cdot [\frac{9}{2}r^2 - \frac{1}{4}r^4] \Big|_0^3 = \frac{1}{4} \cdot 2\pi \cdot (\frac{81}{2} - \frac{81}{4}) = \frac{81\pi}{8}. \quad \text{(2 pts)}$$