1091模組13-17班 微積分2 期考解答和評分標準

1. (6%) Find $f(x)$ so that $\int_0^x f(t)dt = \int_{\sin x}^{x^3} e^{-t^2}\, dt$.

Solution:

By the Fundamental Theorem of Calculus Part 1 (FTC#1), we know that

$$\frac{d}{dx}\int_0^x f(t)\ dt = f(x)$$

The right hand side of the given equation can be written as

$$\int_{\sin x}^{x^3} e^{-t^2}\ dt = \int_0^{x^3} e^{-t^2}\ dt - \int_0^{\sin x} e^{-t^2}\ dt$$

By Chain Rule we can find the derivative of the right hand side

$$\frac{d}{dx}\left[\int_0^{x^3} e^{-t^2}\ dt - \int_0^{\sin x} e^{-t^2}\ dt\right] = e^{-(x^3)^2}(3x^2) - e^{-(\sin x)^2}(\cos x)$$

Simplify to get our conclusion

$$f(x) = 3x^2 e^{-x^6} - e^{-\sin^2 x}\cos x$$

□

- Grading:

This problem is designed to see if the students know how to use Part 1 of the Fundamental Theorem of Calculus.

(1%) for the left side of the equation. (5%) for the right side of the equation.

List of expected mistakes:

- $\frac{d}{dx}\int_0^x f(t)\ dt = f(t)$ is -1%
- $\frac{d}{dx}\int_0^x f(t)\ dt = f(x) - f(0)$ is -1%
- Missing negative sign is -1%
- Didn't apply Chain Rule is -4%
- Taking the derivative of x^3 and $\sin x$ but got it wrong, -1% each
- $e^{-x^2}, e^{-(3x^2)^2}$, and $e^{-(\cos x)^2}$ are -2% each

2. Evaluate the following integrals.

(a) (5%) $\int \tan x \sec^4 x\, dx$.

(b) (8%) $\int \frac{3x+2}{x(x^2+2x+2)}\, dx$.

(c) (7%) $\int_0^4 \frac{x^3+1}{\sqrt{16-x^2}}\, dx$

Solution:

(a) **Solution 1:**

$$\int \tan x \sec^4 x\, dx = \int \tan x \cdot \sec^2 x \sec^2 x\, dx = \int \tan x(\tan x + 1)\sec^2 x\, dx \qquad \text{(1 pt)}$$

$$\overset{u=\tan x}{\underset{du=\sec^2 x dx}{=\!=\!=}} \int u(u^2+1)du \qquad \text{(2 pts for substitution)}$$

$$= \frac{1}{4}u^4 + \frac{1}{2}u^2 + c = \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + c \qquad \text{(2 pts)}$$

Solution 2:

$$\int \tan x \sec^4 x\, dx = \int \sec^3 x \sec x \tan x\, dx \qquad \text{(1 pt)}$$

$$\overset{u=\sec x}{\underset{du=\sec x \tan x dx}{=\!=\!=}} \int u^3\, du = \frac{1}{4}u^4 + c = \frac{1}{4}\sec^4 x + c \qquad \text{(2 pts)}$$

(b) $\frac{3x+2}{x(x^2+2x+2)} = \frac{a}{x} + \frac{bx+c}{x^2+2x+2}$ (1 pt)

$\Rightarrow a = 1,\ b = -1,\ c = 1$

$\frac{3x+2}{x(x^2+2x+2)} = \frac{1}{x} + \frac{-x+1}{x^2+2x+2}$ (2 pts)

$$\int \frac{3x+2}{x(x^2+2x+2)}dx = \int \frac{1}{x} + \frac{-x+1}{x^2+2x+2}dx$$

$$= \ln|x| + \int \frac{-x+1}{x^2+2x+2}dx \qquad \left(\text{1 pt for } \int \frac{1}{x} = \ln|x|\right)$$

$$\overset{u=x+1}{\underset{du=dx}{=\!=\!=}} \ln|x| + \int \frac{-u+2}{u^2+1}du \qquad \text{(2 pts for completing the square and substitution)}$$

$$= \ln|x| - \frac{1}{2}\ln(u^2+1) + 2\tan^{-1} u + c \qquad \left(\text{1 pt for } \int \frac{u}{u^2+1}du\right.$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) + 2\tan^{-1}(x+1) + c \qquad \left.\text{1 pt for } \int \frac{1}{u^2+1}du\right)$$

(c)

$$\int_0^4 \frac{x^3+1}{\sqrt{16-x^2}}dx \overset{x=4\sin\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}}{\underset{dx=4\cos\theta d\theta}{=\!=\!=}} \int_0^{\frac{\pi}{2}} \frac{64\sin^3\theta + 1}{4\cos\theta} 4\cos\theta\, d\theta$$

(4 pts. 1 pt for $x = 4\sin\theta$. 2 pts for integrand and differentials.
1 pt for the upper and lower bound)

$$= \int_0^{\frac{\pi}{2}} 64\sin^3\theta + 1\, d\theta$$

$$= 64\int_0^{\frac{\pi}{2}} \sin^2\theta \sin\theta\, d\theta + \frac{\pi}{2} \qquad \left(\text{1 pt for } \int_0^{\frac{\pi}{2}} 1\, d\theta\right)$$

$$\overset{u=\cos\theta}{=\!=\!=} 64\int_1^0 (1-u^2)(-du) + \frac{\pi}{2}$$

$$= \frac{128}{3} + \frac{\pi}{2} \qquad \left(\text{2 pts for } \int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta\right)$$

3. Determine whether the integral is convergent or divergent. Evaluate convergent integral(s).

(a) (5%) $\int_0^\infty \frac{dx}{x^2+a^2}$, where $a > 0$ is a constant.

(b) (5%) $\int_0^\infty \frac{dx}{x^2}$.

(c) (5%) $\int_0^\infty \frac{dx}{x^2-a^2}$, where $a > 0$ is a constant.

Solution:

(a)

$$\int_0^t \frac{dx}{x^2+a^2} = \frac{1}{a}\int_0^t \frac{1}{1+(\frac{x}{a})^2}\frac{dx}{a}$$

$$\xlongequal[du=\frac{1}{a}dx]{u=\frac{x}{a}} \frac{1}{a}\int_0^{\frac{t}{a}} \frac{1}{1+u^2}du = \frac{1}{a}\tan^{-1}(\frac{t}{a}) \qquad \text{(3 pts)}$$

$$\int_0^\infty \frac{dx}{x^2+a^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+a^2} = \lim_{t\to\infty}\frac{1}{a}\tan^{-1}(\frac{t}{a}) = \frac{\pi}{2a} \qquad \text{(2 pts)}$$

Hence the improper integral converges and the value is $\frac{\pi}{2a}$.

(b)

$$\int_0^\infty \frac{1}{x^2}dx = \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2} \qquad \text{(2 pts for decomposing it into two improper integrals)}$$

$$\because \int_0^1 \frac{1}{x^2}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}dx = \lim_{t\to 0^+}(\frac{1}{t}-1) = \infty \text{ diverges}$$

$$\therefore \int_0^\infty \frac{1}{x^2}dx \text{ diverges} \qquad \text{(2 pts for the divergence of } \int_0^1 \frac{dx}{x^2}\text{. 1 pt for the conclusion)}$$

(c)

$$\frac{1}{x^2-a^2} \to -\infty \text{ as } x \to a^- \qquad \text{(1 pt)}$$

$$\int_0^a \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int_0^a \frac{1}{x-a} - \frac{1}{x+a}dx = \frac{1}{2a}\lim_{t\to a^-}\int_0^t \frac{1}{x-a} - \frac{1}{x+a}dx = \frac{1}{2a}\lim_{t\to a^-}\ln\left|\frac{t-a}{t+a}\right| = -\infty.$$

Hence $\int_0^a \frac{1}{x^2-a^2}dx$ diverges. (3 pts)

Therefore $\int_0^\infty \frac{dx}{x^2-a^2} = \int_0^a \frac{dx}{x^2-a^2} + \int_a^{2a} \frac{dx}{x^2-a^2} + \int_{2a}^\infty \frac{dx}{x^2-a^2}$ diverges (1 pt)

4. Let S be the region enclosed by $y = \ln x$, $y = 1$, and $x = 4$. Let $\mathbf{R}$ be the solid of the revolution by rotating S around the x-axis.

   (a) (7%) Find the volume of $\mathbf{R}$ by the disc method.

   (b) (7%) Find the volume of $\mathbf{R}$ by the cylindrical shell method.

Solution:

(a) We use the disc method. We first evaluate

$$\begin{aligned}
\tilde{V} =& \pi \int_e^4 (\ln x)^2 dx = \pi \int_e^4 \ln x d(x \ln x - x) \text{ (2\% for the correct integral)} \\
=& \pi(x \ln x - x) \ln x|_e^4 - \pi \int_e^4 (x \ln x - x)(\frac{1}{x}) dx \\
=& \pi(x \ln x - x) \ln x|_e^4 - \pi[x \ln x - x - x]|_e^4 \\
=& \pi(x(\ln x)^2 - 2x \ln x + 2x)|_e^4 \text{ (2 \% for correct antiderivative)} \\
=& \pi(4(\ln 4)^2 - 8 \ln 4 + 8) - \pi(e - 2e + 2e) = 16\pi((\ln 2)^2 - \ln 2) + \pi(8 - e). \text{ (2\%)}
\end{aligned}$$

So the required volume is

$$V = \tilde{V} - \pi(4 - e) = 16\pi((\ln 2)^2 - \ln 2) + 4\pi. \text{ (1\%)}$$

Some people may do the following way:

$$\begin{aligned}
\int (\ln x)^2 dx =& x(\ln x)^2 - \int x d(\ln x)^2 \\
=& x(\ln x)^2 - 2 \int x (\ln x)(\frac{1}{x}) dx \\
=& x(\ln x)^2 - 2(x \ln x - x) = x(\ln x)^2 - 2x \ln x + 2x.
\end{aligned}$$

(b) Now we use the cylindrical shell method. Here we write $x = e^y$ and the domain of integration becomes $y \in [1, \ln 4]$. So the volume is

$$\begin{aligned}
V =& 2\pi \int_1^{\ln 4} y[4 - e^y] dy = 2\pi(2y^2 - ye^y + e^y)|_1^{\ln 4} \\
& \text{(3\% for the correct integral, 2\% for the correct antiderivative)} \\
=& 2\pi[(2(\ln 4)^2 - 4 \ln 4 + 4) - (2 - e + e)] \\
=& 16\pi((\ln 2)^2 - \ln 2) + 4\pi. \text{ (2\%)}
\end{aligned}$$

5. The consumer's surplus CS is obtained from the demand curve $p = D(q) = \dfrac{1000}{(0.2q+1)^3}$ and the current unit price p with the formula

$$CS(p) = \left[\int_0^q D(x)\,dx\right] - pq = \left[\int_0^{D^{-1}(p)} D(x)\,dx\right] - pD^{-1}(p)$$

where $q = D^{-1}(p)$ is the quantity demanded obtained from $p = D(q)$.

(a) (4%) Compute $D^{-1}(p)$.

(b) (6%) Find the consumer surplus for the demand function $D(x)$ when $p = 8$.

(c) (5%) Compute $CS'(p)$.

(d) (3%) Explain why $CS(p)$ is a decreasing function on its domain.

Solution:

(a) $p = \dfrac{1000}{(0.2q+1)^3}$, $\quad (0.2q+1)^3 = \dfrac{1000}{p}$, $\quad 0.2q = \sqrt[3]{\dfrac{1000}{p}} - 1$, $\quad q = 5\sqrt[3]{\dfrac{1000}{p}} - 5$

$$D^{-1}(p) = 50p^{-1/3} - 5$$

☐

(b) When $p = 8$, $q = 50(8^{-1/3}) - 5 = 20$

$$CS(8) = \int_0^{20} \frac{1000}{(0.2x+1)^3}\,dx - 8\cdot 20 = \left[-2500(0.2x+1)^{-2}\right]_0^{20} - 160 = -100 + 2500 - 160 = 2240$$

☐

(c)

$$CS(p) = \int_0^{50p^{-1/3}-5} \frac{1000}{(0.2x+1)^3}\,dx - p\left(50p^{-1/3} - 5\right)$$

$$CS'(p) = p\left(\frac{-50}{3}p^{-4/3}\right) - \left(50p^{-1/3} - 5\right) + p\left(\frac{-50}{3}p^{-4/3}\right)$$

$$CS'(p) = -\left(50p^{-1/3} - 5\right)$$

☐

or

$$CS(p) = \int_0^{50p^{-1/3}-5} \frac{1000}{(0.2x+1)^3}\,dx - p\left(50p^{-1/3} - 5\right)$$

$$= \left[-2500(0.2x+1)^{-2}\right]_0^{50p^{-1/3}-5} - p\left(50p^{-1/3} - 5\right) = -25p^{2/3} - p\left(50p^{-1/3} - 5\right)$$

$$CS'(p) = \left(\frac{-50}{3}p^{-1/3}\right) - \left(50p^{-1/3} - 5\right) + p\left(\frac{-50}{3}p^{-4/3}\right)$$

$$CS'(p) = -\left(50p^{-1/3} - 5\right)$$

☐

(d) From (c), CS is a decreasing functions when $-(50p^{-1/3} - 5) < 0$, in other words, $p < 1000$. Notice that when $p = 1000$. $q = D^{-1}(1000) = 0$. Therefore $p \le 1000$ is the domain and $CS(p)$ is decreasing on its domain.

☐

- Grading:

  (a) Finding the inverse (4%). Algebra mistakes are -1% each. Any mistake here should change their answer in (b) and (c).

  (b) Find corresponding q (2%) and evaluate the integral (4%). Full credit if they got (a) wrong but found correct q and integrated correctly using their answer. Algebra mistakes -1% each, integral mistakes -2% each.

  (c) They can use FTC or just integrate first, then derivative. (3%) for the derivative of the integral part. (2%) for the derivative of pq. Algebra mistakes -1% each, other bigger mistakes -2% each.

  (d) They can explain using the answer from (c) or just explain with words and/or graph (3%). They don't need to find the domain. Gaps in argument is -1% and wrong argument is -2%.

  Expected mistakes:

  - I expect a lot of people to have trouble understanding the problem.
  - I expect a lot of mistakes in (c)
  - I expect a lot of answers in (d) that don't use (c)
  - When grading, award points for anything in the right direction

6. Solve the following differential equations.

(a) (7%) $(1+x^2)y' + 2xy^2 = 0$, $y(0) = 1$.

(b) (8%) $(1+x^2)y' + xy = x(1+x^2)$, $y(0) = 1$.

Solution:

(a) We have

$$\frac{1}{y^2}y' = \frac{-2x}{1+x^2}, \quad (2\%)$$

hence

$$\frac{1}{y} = \ln(1+x^2) + C \quad (2\%), \quad y = \frac{1}{\ln(1+x^2)+C} \quad (1\%).$$

Now we have

$$1 = y(0) = \frac{1}{\ln 1 + C}, \quad C = 1, \quad (2\%)$$

so

$$y = \frac{1}{\ln(1+x^2)+1}.$$

(b) We have

$$y' + \frac{x}{1+x^2}y = x,$$

so $P(x) = \frac{x}{1+x^2}$, $Q(x) = x$, and

$$I(x) = e^{\int P(x)dx} = e^{\frac{1}{2}\ln(1+x^2)} = \sqrt{1+x^2}. \quad (3\%)$$

Thus

$$\begin{aligned} y &= \frac{1}{I(x)} \int I(x)Q(x)dx \quad (1\%) \\ &= \frac{1}{\sqrt{1+x^2}} \int x\sqrt{1+x^2}dx \\ &= \frac{1}{3}(1+x^2) + \frac{C}{\sqrt{1+x^2}}. \quad (2\%) \end{aligned}$$

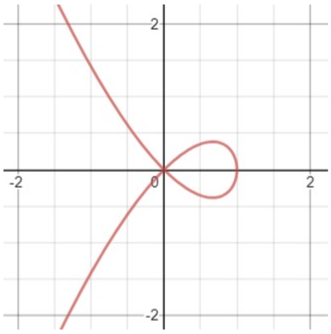
Now

$$1 = y(0) = \frac{1}{3} + C, \quad C = \frac{2}{3}. \quad (2\%)$$

Hence

$$y = \frac{1}{3}\left(1 + x^2 + \frac{2}{\sqrt{1+x^2}}\right).$$

7. Consider the parametric curve $(x(t), y(t)) = (1 - t^2, t - t^3)$.



(a) (2%) The curve passes through the origin twice. Find t_1 and t_2 so that

$$(x(t_1), y(t_1)) = (x(t_2), y(t_2)) = (0, 0).$$

(b) (4%) Find the tangent lines of the curve at $(0, 0)$.

(c) (6%) $(x(t), y(t)) = (1 - t^2, t - t^3)$, $t_1 \le t \le t_2$, forms a loop. Find the area enclosed by the loop.

Solution:

(a) $t_1 = -1$ (1%), $t_2 = 1$ (1%).

(b)

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{1 - 3t^2}{-2t}, \quad (2\%)$$

so the tangent line is $y = -x$ when $t = -1$ (1%), and $y = x$ when $t = 1$ (1%). These are two tangent lines of the curve at $(0, 0)$.

(c)

$$\begin{aligned} \text{Area} &= \left| \int_{-1}^{1} y(t)x'(t)dt \right| \quad (2\%) \\ &= \left| \int_{-1}^{1} (t - t^3)(-2t)dt \right| \quad (1\%) \\ &= \left| -2 \int_{-1}^{1} t^2 - t^4 dt \right| \\ &= 2\left(\frac{1}{3}t^3 - \frac{1}{5}t^5\right)\Big|_{-1}^{1} \quad (2\%) \\ &= \frac{8}{15}. \quad (1\%) \end{aligned}$$