1091模組06-12班 微積分2 期考解答和評分標準

1. (8 pts) Let f be a continuous function on $\mathbb{R}$ such that

$$\int_0^{x^3} f(t)\,\mathrm{d}t = x^3 \cdot \cos(\pi x) \quad \text{for all } x.$$

Find $f(1)$.

Solution:

Since f is continuous on $\mathbb{R}$, by Fundamental Theorem of Calculus **(2%)**, we have

$$\frac{d}{dx}\int_0^{x^3} f(t)\,dt = \frac{d}{dx}\left(x^3 \cdot \cos(\pi x)\right)$$
$$\Rightarrow \quad f(x^3) \cdot 3x^2 \ \textbf{(2\%)} = 3x^2 \cdot \cos(\pi x) - \pi x^3 \cdot \sin(\pi x) \ \textbf{(1\%)}$$

To find $f(1)$, we solve $x^3 = 1$ to obtain that $x = 1$ **(1%)**. For $x = 1$, we have

$$f(1) \cdot 3 = 3 \cdot \cos(\pi) - \pi \cdot \sin(\pi) = -3 \ \Rightarrow f(1) = -1 \ \textbf{(2\%)}.$$

2. (16 pts) Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{\sqrt{x}\,dx}{(1+\sqrt{x})^4}$.

(b) $\displaystyle\int_0^1 \sin^{-1}(\sqrt{x})\,dx$.

Solution:

Answer: Must show clearly the steps of substitution and integration by parts

(a) Method-1:

- (2%) Set $y = \sqrt{x} \Rightarrow dx = 2ydy \Rightarrow \int_0^1 \frac{\sqrt{x}}{(1+\sqrt{x})^4}dx = \int_0^1 \frac{2y^2}{(1+y)^4}dy$

  (1%) $= 2\int_0^1 \frac{1-(1-y^2)}{(1+y)^4}dy = 2\int_0^1 [\frac{1}{(1+y)^4} - \frac{(1-y)}{(1+y)^3}]dy$

  (2%) $= 2\int_0^1 [\frac{1}{(1+y)^4} - \frac{2-(1+y)}{(1+y)^3}]dy \; = 2\int_0^1 [\frac{1}{(1+y)^4} - \frac{2}{(1+y)^3} + \frac{1}{(1+y)^2}]dy$

  (2%) $= -2[\frac{1}{3(1+y)^3} - \frac{1}{(1+y)^2} + \frac{1}{(1+y)}]_0^1 \; = \; \frac{1}{12} \; \cdots$

Method-2:

- (2%) Set $y = 1 + \sqrt{x} \Rightarrow dx = 2(y-1)dy$

  (3%) $\Rightarrow \int_0^1 \frac{\sqrt{x}}{(1+\sqrt{x})^4}dx = \int_1^2 \frac{2(y-1)^2}{y^4}dy \; = 2\int_1^2 [\frac{1}{y^2} - \frac{2}{y^3} + \frac{1}{y^4}]dy$

  (2%) $= -2[\frac{1}{y} - \frac{1}{y^2} + \frac{1}{3y^3}]_1^2 \; = \; \frac{1}{12} \; \cdots$

(b) Method-1:

- (2%) Set $y = \sqrt{x} \Rightarrow dx = 2ydy \Rightarrow \int_0^1 \sin^{-1}(\sqrt{x})dx = \int_0^1 \sin^{-1}(y)\,2ydy$

  (2%) $= \int_0^1 \sin^{-1}(y)\,(y^2)'dy \; = [y^2\sin^{-1}(y)]_0^1 - \int_0^1 \frac{y^2}{\sqrt{1-y^2}}dy$

  (2%) $= \frac{\pi}{2} - \int_0^1 \frac{1-(1-y^2)}{\sqrt{1-y^2}}dy \; = \frac{\pi}{2} - \int_0^1 [\frac{1}{\sqrt{1-y^2}} + \sqrt{1-y^2}]dy$

  (2%) $= \frac{\pi}{2} - [\sin^{-1}(y)]_0^1 + \int_0^{\pi/2}\cos^2(\theta)d\theta$ where $y = \sin\theta$

  (1%) $= \frac{1}{2}[\theta + \frac{\sin(2\theta)}{2}]_0^{\pi/2} \; = \; \frac{\pi}{4} \; \cdots$

Method-2:

- (4%) Set $y = \sin^{-1}(\sqrt{x}) \Rightarrow x = \sin^2(y)$ and $dx = \sin(2y)dy$

  (2%) $\Rightarrow \int_0^1 \sin^{-1}(\sqrt{x})dx = \int_0^{\pi/2} y\,\sin(2y)dy$

  (2%) $= \int_0^{\pi/2} y\,(-\frac{\cos(2y)}{2})'dy \; = \frac{-1}{2}[y\,\cos(2y)]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos(2y)dy$

  (1%) $= \frac{\pi}{4} + \frac{1}{4}[\sin(2y)]_0^{\pi/2} \; = \; \frac{\pi}{4} \; \cdots$

3. (13 pts)

(a) Decompose $\dfrac{x^2+4x+5}{(x+1)^2(x^2+2x+3)}$ into partial fractions.

(b) Evaluate the indefinite integral $\displaystyle\int \frac{x^2+4x+5}{(x+1)^2(x^2+2x+3)}\mathrm{d}x$.

Solution:

(a) Let

$$\frac{x^2+4x+5}{(x+1)^2(x^2+2x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+3}. \quad \text{(2 pts)}$$

By clearing the denominator we have

$$x^2+4x+5 = A(x+1)(x^2+2x+3) + B(x^2+2x+3) + (Cx+D)(x+1)^2. \quad \text{(1 pt)}$$

By comparing the coefficients, we have

$$\begin{cases} A & & +C & & = 0, \\ 3A & +B & +2C & +D & = 1, \\ 5A & +2B & +C & +2D & = 4, \\ 3A & +3B & & +D & = 5, \end{cases} \quad \text{(4 pts)} \quad \Rightarrow \begin{cases} A & = 1, \\ B & = 1, \\ C & = -1, \\ D & = -1. \end{cases} \quad \text{(1 pts)}$$

Therefore, we get

$$\frac{x^2+4x+5}{(x+1)^2(x^2+2x+3)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{-x-1}{x^2+2x+3}. \quad \text{(1 pt)}$$

(b) By (a) we have the indefinite integral

$$\frac{x^2+4x+5}{(x+1)^2(x^2+2x+3)}dx = \int [\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{x+1}{x^2+2x+3}]dx \quad \text{(1 pt)}$$

$$= \ln|x+1| - \frac{1}{x+1} - \frac{1}{2}\ln(x^2+2x+3) + C. \quad \text{(3 pts)}$$

4. (16 pts)

(a) Find the orthogonal trajectories of the family of curves $y = K \cdot \tan^2 x$, where K is an arbitrary constant.

(b) Solve the initial value problem : $x(x+1)\frac{dy}{dx} + y = (x+1)^2 \sin x \cos x, \quad y\left(\frac{\pi}{2}\right) = 0.$

Solution:

(a) (i) The slope of each point on the family of curves is

$$\frac{dy}{dx} = K \cdot 2\tan x \cdot \sec^2 x \quad \text{(2 pts)} \quad = \frac{y}{\tan^2 x} \cdot 2\tan x \cdot \sec^2 x$$
$$= 2y \cdot \sec^2 x \cdot \cot x = \frac{2y}{\sin x \cdot \cos x}. \quad \text{(2 pts)}$$

(ii) Solve $\frac{dy}{dx} = \frac{-\sin x \cdot \cos x}{2y}$. (2 pts)

We have

$$\frac{dy}{dx} = \frac{-\sin x \cdot \cos x}{2y}$$
$$\Rightarrow \int 2y dy = \int (-\sin x \cdot \cos x) dx = \frac{-1}{2} \int \sin(2x) dx$$
$$\Rightarrow y^2 = \frac{\cos(2x)}{4} + C. \quad \text{(2 pts)}$$

The orthogonal trajectories of the family of curves $y^2 = \frac{\cos(2x)}{4} + C$.

(b) The standard form of the differential equation is

$$\frac{dy}{dx} + \frac{1}{x(x+1)} \cdot y = \frac{x+1}{x} \cdot \sin x \cdot \cos x.$$

Since

$$e^{\int \frac{1}{x(x+1)} dx} = e^{\int (\frac{1}{x} - \frac{1}{x+1}) dx} = e^{\ln |\frac{x}{x+1}|} = |\frac{x}{x+1}|,$$

we can take the integration factor as $I(x) = \frac{x}{x+1}$. (3 pts)

Then we have

$$I(x) \cdot [\frac{dy}{dx} + \frac{1}{x(x+1)} \cdot y] = I(x) \cdot [\frac{x+1}{x} \cdot \sin x \cdot \cos x]$$
$$\Rightarrow [\frac{x}{x+1} \cdot y]' = [I(x) \cdot y]' = \frac{1}{2} \sin(2x)$$
$$\Rightarrow \frac{x}{x+1} \cdot y = I(x) \cdot y = \int \frac{1}{2} \sin(2x) dx = \frac{-1}{4} \cos(2x) + C$$
$$\Rightarrow y = \frac{x+1}{x} \cdot [\frac{-1}{4} \cos(2x) + C]. \quad \text{(3 pts)}$$

Since $y(\frac{\pi}{2}) = 0$, we get $\frac{-1}{4} = C$. So the solution is $y = \frac{x+1}{x} \cdot [\frac{-1}{4} \cos(2x) - \frac{1}{4}]$. (2 pts)

5. (14 pts) Let C be a curve whose parametrisation is given by $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$ with $0 \le t \le \pi$.

(a) Find the arclength of C.

(b) Find $\dfrac{dy}{dx}$ in terms of t and find the point Q in x-y coordinates at which the tangent to C is perpendicular to the x-axis.

Solution:

(a)

$$\frac{dx}{dt} = e^t(\cos t - \sin t)$$
$$\frac{dy}{dt} = e^t(\sin t + \cos t).$$

Arc length is then computed by the formula

$$\int_0^\pi \sqrt{(\frac{dx}{dt})^2 + (\frac{dx}{dt})^2}\ dt = \int_0^\pi \sqrt{2}e^t\ dt = \sqrt{2}(e^\pi - 1).$$

Grading Guideline. *Arc length formula (including lower and upper limits) 3pt*
differentiation computation 2pt
integral computation 2pt.

(b) With the computation above

$$\frac{dy}{dx} = \frac{\sin t + \cos t}{\cos t - \sin t}.$$

Vertical tangents occur, if exists, when $\cos t - \sin t = 0$, that is $t = \frac{\pi}{4}$. We need to check if it is indeed a vertical tangent. We can either compute

$$\lim_{t \to \frac{\pi}{4}}^{\pm} \frac{dy}{dx} = \pm\infty$$

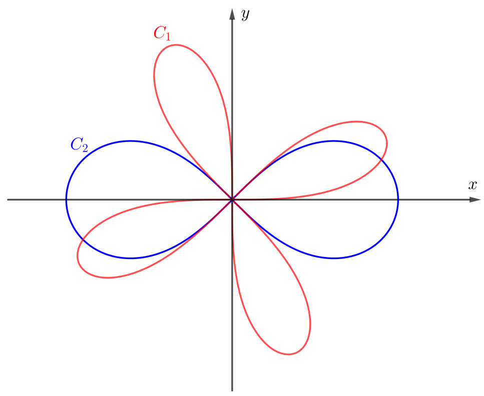
or just say the numerator is non-zero. Finally, converting it into cartesian coordinates, we get $Q = e^{\frac{\pi}{4}}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Grading Guideline. $\frac{dy}{dx}$ *(2pt= chain rule (1pt) + computation (1pt))*
vertical tangent (3pt)=denominator=0 (1pt)+t-value(1pt)+limit(1pt)
converting to cartesian coordinates (1pt).

6. (10 pts) Consider two polar curves C_1 and C_2 defined by

$$C_1 : r^2 = 4\sin(4\theta), \qquad C_2 : r^2 = 4\cos(2\theta).$$

Find the area of the region that lies inside both C_1 and C_2.



Solution:

By symmetry, we compute the intersection of curves C_1 and C_2 when $0 \le \theta \le \frac{\pi}{4}$.

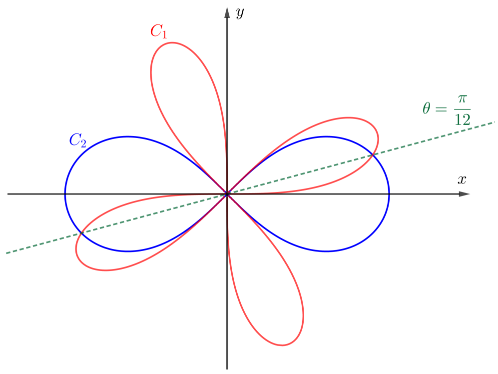
$$\begin{cases} r^2 = 4\sin(4\theta) \\ r^2 = 4\cos(2\theta), \end{cases} \Rightarrow 4\cos(2\theta)\cdot(\sin(2\theta) - 1) = 0 \quad \text{(2 pts)}$$

$$\Rightarrow \cos(2\theta) = 0 \text{ or } \sin(2\theta) = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{2} \text{ or } \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{\pi}{12}. \quad \text{(3 pts)}$$

So we have

$$\begin{aligned} &\text{Area of the region inside } C_1 \text{ and } C_2 \\ &= 2\cdot\left[\int_0^{\frac{\pi}{12}} \frac{1}{2}\cdot 4\sin(4\theta)d\theta + \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2}\cdot 4\cos(2\theta)d\theta\right] \quad \text{(3 pts)} \\ &= (-\cos(4\theta))\Big|_0^{\frac{\pi}{12}} + (2\sin(2\theta))\Big|_{\frac{\pi}{12}}^{\frac{\pi}{4}} = (-\frac{1}{2}+1) + 2(1-\frac{1}{2}) = \frac{3}{2}. \quad \text{(2 pts)} \end{aligned}$$



7. (10 pts) The ellipse

$$x^2 + \frac{y^2}{5} = 1$$

is rotated about the x-axis to form a surface called an *oblate spheroid*. Find the surface area of this oblate spheroid.

Solution:

Marking scheme.
(2M) *Evaluation of ds
(2M) ** Integrand
(1M) Integration limits
(2M) ***Convert the integral into $C \cdot \int \sec^3\theta d\theta$ by a suitable substitution
(2M) ****Correct evaluation of $\int \sec^3\theta d\theta$
(1M) Correct answer

Partial credits.
(*) 1M is awarded as long as the candidate attempts to compute ds
(**) Here a candidate will receive at most 1M for

- making mistakes up to a constant factor
- mentioning $\int 2\pi y \cdot ds$ (or any equivalent form)

(***) 1M is taken away if a student makes mistakes in the integration limits after conducting a substitution
(****) No derivation is required. At most 1M can be awarded to a candidate with an incorrect evaluation who (1) attempts to derive $\int \sec^3\theta d\theta$ or (2) makes minor/obvious typos.

Sample Solution 1.
One can obtain an ellipsoid by using the upper half of the ellipse $y = \sqrt{5(1-x^2)}$ for which $\frac{dy}{dx} = \sqrt{5} \cdot \frac{-x}{\sqrt{1-x^2}}$ and

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{5x^2}{1-x^2}} = \underbrace{\sqrt{\frac{1+4x^2}{1-x^2}}}_{(2M)}$$

By symmetry, the surface area is given by

$$\underbrace{2 \cdot \int_0^1 2\pi \underbrace{\sqrt{5(1-x^2)}}_{y} \underbrace{\sqrt{\frac{1+4x^2}{1-x^2}} dx}_{ds}}_{(3M)} = 4\sqrt{5}\pi \int_0^1 \sqrt{1+4x^2} dx$$

To evaluate $\int \sqrt{1+4x^2} dx$, we substitute $x = \frac{1}{2}\tan\theta$, then

$$\begin{aligned} \int \sqrt{1+4x^2} dx &= \frac{1}{2} \int \sec^3\theta d\theta \qquad \text{(2M)} \\ &= \frac{1}{4}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C \qquad \text{(2M)} \\ &= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln(2x + \sqrt{1+4x^2})\right) + C \end{aligned}$$

Hence, the surface area equals to $\sqrt{5}\pi(2\sqrt{5} + \ln(2 + \sqrt{5}))$. (1M)

Sample Solution 2 (Parametric way).
The ellipse can be parametrised by $x = \cos t$ and $y = \sqrt{5}\sin t$.

Then $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \underbrace{\sqrt{\sin^2 t + 5\cos^2 t} = \sqrt{1 + 4\cos^2 t}}_{(2M)}.$

By symmetry, the surface area thus equals to

$$\underbrace{2 \cdot \int_0^{\pi/2} 2\pi \underbrace{\sqrt{5}\sin t}_{y} \underbrace{\sqrt{1 + 4\cos^2 t}dt}_{ds}}_{(3M)} = 4\sqrt{5}\pi \int_0^{\pi/2} \sin t\sqrt{1 + 4\cos^2 t}dt$$

To evaluate $\int \sin t\sqrt{1 + 4\cos^2 t}dt$, we let $2\cos(t) = \tan\theta$

$$\begin{aligned}\int \sin t\sqrt{1 + 4\cos^2 t}dt &= -\frac{1}{2}\int \sec^3\theta d\theta \qquad \text{(2M)}\\ &= -\frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C \qquad \text{(2M)}\\ &= -\frac{1}{4}\left(2\cos t\sqrt{1 + 4\cos^2 t} + \ln(2\cos t + \sqrt{1 + 4\cos^2 t})\right) + C\end{aligned}$$

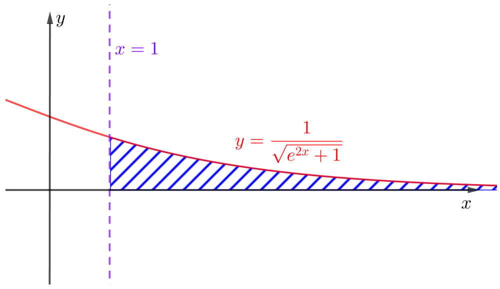
Hence, the surface area equals to $\sqrt{5}\pi(2\sqrt{5} + \ln(2 + \sqrt{5}))$. (1M)

Alternative form of the final answer : $10\pi + \sqrt{5}\pi\ln(2 + \sqrt{5})$.

8. (13 pts) Consider the region bounded by the curve $y = \dfrac{1}{\sqrt{e^{2x}+1}}$, the x-axis and $x = 1$.

(a) Find the volume of the solid obtained by revolving the region about the x-axis.

(b) Student A claims that '*The volume of the solid obtained by revolving the region about the y-axis is also finite.*'. Give a proof to Student A's claim by using the comparison test for improper integrals.



Solution:

(a)

Marking scheme.
(2M) *Integrand
(1M) Integration limits
(1M) Writing down the definition of improper integrals
(3M) **Using a suitable substitution to evaluate the integral
(1M) Correct answer

Partial credits.
(*) At most 1M is taken away for any missing/extra scalar factors.
(**) A candidate will receive

- 2M for having done a correct substitution but incomplete evaluation of the integral
- 3M as long as he/she integrates $\int_a^b \frac{1}{e^{2x}+1} dx$ correctly - does not matter his/her choice of integration limits.

Sample solution.
By disk method, the volume of the solid equals to

$$\underbrace{\int_1^\infty \pi \cdot \frac{1}{e^{2x}+1} dx}_{(2+1M)} = \underbrace{\lim_{t\to\infty} \int_1^t \pi \cdot \frac{1}{e^{2x}+1} dx}_{(1M)}.$$

To evaluate $\int \dfrac{1}{e^{2x}+1} dx$, we let $u = e^{2x}$. Then

$$\int \frac{1}{e^{2x}+1} dx = \underbrace{\frac{1}{2}\int \frac{du}{u(u+1)} = \frac{1}{2}\int \frac{1}{u} - \frac{1}{u+1} du = \frac{1}{2}\ln\left|\frac{u}{u+1}\right| + C = \frac{1}{2}\ln\left|\frac{e^{2x}}{e^{2x}+1}\right| + C}_{(3M)}.$$

Hence, the volume equals to

$$\lim_{t\to\infty} \frac{\pi}{2}\left(\ln\frac{e^{2t}}{e^{2t}+1} - \ln\frac{e^2}{e^2+1}\right) = \underbrace{\pi\left(\frac{1}{2}\ln(e^2+1) - 1\right)}_{(1M)}.$$

Alternative form of the final answer : $-\frac{\pi}{2}\ln\frac{e^2}{e^2+1} = \frac{\pi}{2}\ln\frac{e^2+1}{e^2} = \pi\ln\sqrt{\frac{e^2+1}{e^2}}$.

(b)

Marking scheme.
(2M) *Setting up the correct integral
(1M) Any correct upper bound for $\frac{x}{\sqrt{e^{2x}+1}}$
(2M) **Correct argument using the comparison test

Partial credits.
(*) At most 1M is taken away for any missing/extra scalar factors.
(**) 1M can be awarded to candidates with an incorrect upper bound, but with some attempts of an argument using the comparison test.

In an extreme situation that a candidate didn't set up a correct integral (not of the form $C \cdot \int \frac{x}{\sqrt{e^{2x}+1}} dx$) but demonstrated some understandings of the comparison test, at most 1M will be awarded.

Sample solution.

By shell method, the volume of the solid equals to $\int_1^\infty 2\pi x \cdot \frac{1}{\sqrt{e^{2x}+1}} \mathrm{d}x$. (2M)

Since $0 \leq \frac{x}{\sqrt{e^{2x}+1}} \leq \frac{x}{e^x}$ (1M)

and $\int_1^\infty \frac{x}{e^x}\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{x}{e^x}\mathrm{d}x = \lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 2e^{-1}\right) = 2e^{-1}$ is convergent, (2M)

the comparison test implies that $\int_1^\infty \frac{x}{\sqrt{e^{2x}+1}}\mathrm{d}x$ is convergent. This verifies the claim of Student A.

Alternative argument.

By shell method, the volume of the solid equals to $\int_1^\infty 2\pi x \cdot \frac{1}{\sqrt{e^{2x}+1}} \mathrm{d}x$. (2M)

Since $0 \leq \frac{x}{\sqrt{e^{2x}+1}} \leq \frac{1}{x^2}$ for x sufficiently large (1M)

and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ converges as a p-integral with $p > 1$, (2M)

the comparison test implies that $\int_1^\infty \frac{x}{\sqrt{e^{2x}+1}}\mathrm{d}x$ is convergent. This verifies the claim of Student A.

Remark. $1/x^2$ can be replaced by $1/x^p$ for any $p > 1$ in the above argument.