1. (15%) Find the following limits.

(a) (5%) $\lim_{x\to-\infty} \frac{2e^x+5}{3-e^x}$.
(b) (5%) $\lim_{x\to 0^+} \tan^{-1}(1+\ln x)$.
(c) (5%) $\lim_{x\to\infty} \left(\frac{1}{e}+\frac{e}{x}\right)^x$.

**Solution:**

(a) Formally:

$$\lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to-\infty} 2e^x + 5 = 5$$

$$\lim_{x\to-\infty} 3 - e^x = 3 \neq 0$$

$$\lim_{x\to-\infty} \frac{2e^x+5}{3-e^x} = \frac{\lim_{x\to-\infty} 2e^x+5}{\lim_{x\to-\infty} 3-e^x} = \frac{5}{3}$$

□

Short solution:

$$\lim_{x\to-\infty} \frac{2e^x+5}{3-e^x} = \frac{0+5}{3-0} = \frac{5}{3}$$

□

Grading:
- Correct answer (3%)
- Showing $e^x \to 0$ (2%), this can be done informally (e.g. $\to 0$)
- Clearly showing that they read the problem wrong, but work and answer is correct (3%)
- Minor simplification mistakes (-1%)
- L'Hospital's Rule (-5%)

(b) Formally:

$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to 0^+} 1 + \ln x = -\infty$$

Let $y = 1 + \ln x$,

$$\lim_{x\to 0^+} \tan^{-1}(1+\ln x) = \lim_{y\to-\infty} \tan^{-1} y = -\frac{\pi}{2}$$

□

Short solution:

$$\lim_{x\to 0^+} \tan^{-1}(1+\ln x) = -\frac{\pi}{2} \text{ because } \ln x \to -\infty$$

□

Grading:
- Correct answer (3%)
- Showing $\ln x \to -\infty$ (2%), this can be done informally (e.g. $\to -\infty$ or sketch graph)
- If they sketch graph of $\tan^{-1} x$ (and labelling horizontal asymptotes) but didn't finish (1%)
- Clearly showing that they read the problem wrong, but work and answer is correct (3%) for example, $x \to \infty$
- L'Hospital's Rule (-5%)

(c) Formally:

$$\lim_{x\to\infty} \left(\frac{1}{e}+\frac{e}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(\frac{1}{e}+\frac{e}{x}\right)}$$

$$\lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} \left(\frac{1}{e}+\frac{e}{x}\right) = \frac{1}{e}$$

$$\lim_{x\to\infty} \ln\left(\frac{1}{e}+\frac{e}{x}\right) = \ln\frac{1}{e} = -1$$

$$\lim_{x\to\infty} x\ln\left(\frac{1}{e}+\frac{e}{x}\right) = -\infty$$

Let $y = x\ln\left(\frac{1}{e}+\frac{e}{x}\right)$,

$$\lim_{x\to\infty}\left(\frac{1}{e}+\frac{e}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(\frac{1}{e}+\frac{e}{x}\right)} = \lim_{y\to-\infty} e^y = 0$$

□

Short solution:

$$\lim_{x\to\infty}\left(\frac{1}{e}+\frac{e}{x}\right)^x = 0 \text{ because } \frac{1}{e} < 1 \text{ , } a^\infty \text{ is not an indeterminate form if } a < 1.$$

□

Grading:

- The main mistake students make is to take the limit twice. Without any further work that will be (0%)
- Correct steps (2%) they do not need to take the natural logarithm, writing $(1/e)^\infty$ is also acceptable
- Correct reasoning to get the answer (3%)
- L'Hospital's Rule (-4%) they get (1%) if they did the natural logarithm correctly but clearly thought $\ln(1/e) = 0$, no points if they didn't try to evaluate $\ln(1/e)$

2. (15%) $f(x) = \begin{cases} x^{x+1} & , \text{for } x > 0 \\ 0 & , \text{for } x = 0 \\ \frac{2(1-\cos x)}{x} & , \text{for } x < 0 \end{cases}$.

(a) (5%) Compute $\lim_{x\to 0} f(x)$. Is $f(x)$ continuous at $x = 0$?

(b) (5%) Compute $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$. Is $f(x)$ differentiable at $x = 0$?

(c) (5%) Compute $f'(x)$ for $x \neq 0$.

**Solution:**

(a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{x+1}$. $\because \lim_{x\to 0^+} x = 0,\ \lim_{x\to 0^+} x+1 = 1 \therefore \lim_{x\to 0^+} x^{x+1} = 0^1 = 0$ (2 pts)

**Another solution:**

For $x > 0$, $\ln f(x) = (x+1)\ln x$.

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} (x+1)\ln x = -\infty. \text{Hence } \lim_{x\to 0^+} x^{x+1} = e^{-\infty} = 0.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2(1-\cos x)}{x} = \lim_{x\to 0^-} \frac{2(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim_{x\to 0^-} \frac{2\sin^2 x}{x(1+\cos x)}$$
$$= \lim_{x\to 0^-} \left(\frac{2}{1+\cos x} \cdot \frac{\sin x}{x} \cdot \sin x\right) = 1 \times 1 \times 0 = 0 \quad (2pts)$$

**Another solution:**

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2(1-\cos x)}{x} \xlongequal[\text{l'Hospital's Rule}]{\frac{0}{0}} \lim_{x\to 0^-} \frac{2\sin x}{1} = 0.$$

$$\left.\begin{array}{l} \because \lim_{x\to 0^+} f(x) = 0,\ \lim_{x\to 0^-} f(x) = 0 \ \therefore \lim_{x\to 0} f(x) = 0. \\ \lim_{x\to 0} f(x) = 0 = f(0) \Rightarrow f(x) \text{ is continuous at } x = 0. \end{array}\right\} \quad (1pt)$$

(b) $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} x^x$.

$\ln(x^x) = x\ln x$, $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \xlongequal[\text{l'Hospital's Rule}]{\frac{\infty}{\infty}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

Hence $\lim_{x\to 0^+} x^x = e^0 = 1$.

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x^x = 1$. (2 pts)

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{2(1-\cos x)}{x^2} = \lim_{x\to 0^-} \frac{2(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0^-} \frac{2}{1+\cos x} \frac{\sin^2 x}{x^2} = 1$ (2 pts)

**Another solution:**

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{2(1-\cos x)}{x^2} \xlongequal{\frac{0}{0}} \lim_{x\to 0^-} \frac{2\sin x}{2x} = 1$.

Hence $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 1$.

Therefore $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 1$ and $f$ is differentiable at $x = 0$. (1 pt)

(c) For $x > 0$, $f(x) = x^{x+1}$, $\ln(f(x)) = (x+1)\ln x ...(*)$ (1 pt)

$\frac{d}{dx}(*) \Rightarrow \frac{f'(x)}{f(x)} = \ln x + (1+x)\frac{1}{x} = \ln x + \frac{1}{x} + 1 \Rightarrow f'(x) = x^{x+1}(\ln x + \frac{1}{x} + 1)$. (2 pts)

For $x < 0$, $f'(x) = \frac{d}{dx}\left(\frac{2(1-\cos x)}{x}\right) = 2\frac{\sin x \cdot x - 1 + \cos x}{x^2}$ (2 pts)

3. (15%) Compute the following derivatives.

(a) (5%) Let $f(x) = e^{\tan x}$. Compute $f'(x)$ and $f''(x)$.

(b) (5%) $\frac{d}{dx}(2^{x^2} + x^{e^x})$.

(c) (5%) $\frac{d}{dx}(x \sin^{-1} x + \sqrt{1 - x^2})$.

**Solution:**

(a) Let $y = \tan x$. $f'(x) = (e^y)'y' = e^y \sec^2 x = e^{\tan x} \sec^2 x$.

$$\begin{aligned} f''(x) = (f'(x))' &= (e^{\tan x})' \sec^2 x + e^{\tan x} (\sec^2 x)' \\ &= e^{\tan x} \sec^2 x \cdot \sec^2 x + e^{\tan x} \cdot 2 \sec x \cdot \sec x \tan x \\ &= e^{\tan x} \sec^2 x (\sec^2 x + 2 \tan x). \end{aligned}$$

Scoring rules:

$$f'(x) = e^{\tan x} (\tan x)' \quad (2\%), \quad (\tan x)' = \sec^2 x \quad (1\%),$$
$$f''(x) = (e^{\tan x})' \sec^2 x + e^{\tan x} (\sec^2 x)' \quad (1\%), \quad (\sec^2 x)' = 2 \sec^2 x \tan x \quad (1\%).$$

(b)

$$\begin{aligned} \frac{d}{dx}(2^{x^2} + x^{e^x}) &= \frac{d}{dx}\left(e^{x^2 \ln 2} + e^{e^x \ln x}\right) \\ &= e^{x^2 \ln 2} 2 \ln 2 \cdot x + e^{e^x \ln x}\left(e^x \ln x + e^x \frac{1}{x}\right) \\ &= 2 \ln 2 \cdot 2^{x^2} x + x^{e^x} e^x \left(\ln x + \frac{1}{x}\right). \end{aligned}$$

Scoring rules:

$$(2^{x^2} + x^{e^x})' = (2^{x^2})' + (x^{e^x})' \quad (1\%),$$
$$(2^{x^2})' = 2^{x^2} (\ln 2 x^2)' (1\%) = 2 \ln 2 \cdot 2^{x^2} x (1\%),$$
$$(x^{e^x})' = x^{e^x} (e^x \ln x)' (1\%) = x^{e^x} e^x \left(\ln x + \frac{1}{x}\right) (1\%).$$

(c)

$$\begin{aligned} \frac{d}{dx}(x \sin^{-1} x + \sqrt{1 - x^2}) &= \left(\sin^{-1} x + x \frac{1}{\sqrt{1 - x^2}}\right) + \frac{1}{2\sqrt{1 - x^2}}(-2x) \\ &= \sin^{-1} x \end{aligned}$$

Scoring rules:

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}} \quad (2\%), \quad (\sqrt{1 - x^2})' = \frac{1}{2\sqrt{1 - x^2}}(-2x) \quad (1\%),$$
$$(x \sin^{-1} x + \sqrt{1 - x^2})' = (x \sin^{-1} x)' + (\sqrt{1 - x^2})' \quad (1\%),$$
$$(x \sin^{-1} x)' = \left(\sin^{-1} x + x \frac{1}{\sqrt{1 - x^2}}\right) \quad (1\%).$$

4. (12%) Suppose that near the point $(3,8)$, $3y^{2/3}+xy=36$ defines a function $y=f(x)$.

   (a) (4%) Compute $\frac{dy}{dx}$ at $(3,8)$ which is $f'(3)$.

   (b) (3%) Use the linear approximation to estimate $f(3.01)$.

   (c) (5%) Compute $\frac{d^2y}{dx^2}$ at $(3,8)$. Is the estimation from (b) larger than or smaller than $f(3.01)$?

**Solution:**

(a) Consider $y$ as a function of $x$. Differentiate the equation $3(y(x))^{2/3}+x\cdot y(x)=36$ with respect to $x$. We obtain $3\cdot\frac{2}{3}y^{-\frac{1}{3}}\cdot y'+y+x\cdot y'=0....(*)$(3 pts)
(1 pt for trying to differentiate the equation with respect to $x$. 2 pts for correct result.)
At $(3,8)$, $(*)\Rightarrow y'+8+3y'=0\Rightarrow y'=2$. (1 pt)

(b) The linear approximation of $f$ at $x=3$ is

$$\begin{aligned} L(x) &= f(3)+f'(3)\cdot(x-3) && \text{(1 pt)}\\ &= 8-2\cdot(x-3) && \text{(1 pt)}\\ f(3.01)\approx L(3.01) &= 8-0.02=7.98 && \text{(1 pt)} \end{aligned}$$

(c) $\frac{d}{dx}(*)\Rightarrow -\frac{2}{3}y^{-4/3}(y')^2+2y^{-1/3}y''+y'+y'+xy''=0$ (3 pts)
(1 pt for trying to differentiate the equation with respect to $x$. 2 pts for correct result.)
At $(3,8)$, $y'=-2$, $-\frac{2}{3}\times\frac{1}{16}\times 4+y''+(-4)+3y''=0$
$\Rightarrow 4y''=\frac{25}{6}\Rightarrow y''=\frac{25}{24}>0$ (1 pt)
Because $y''(x)$ is continuous near $x=3$ and $y''(3)=\frac{25}{24}>0$, we know that $y''(x)>0$ for $x$ near 3. Hence the curve is concave upward near $(3,8)$ and the linear approximation is smaller than $f(3.01)$. (1 pt)

5. (15%) For each limit, state its indeterminate form and use l'Hospital's rule to compute it.

(a) (5%) $\lim_{x\to 0} \frac{\tan^{-1}(x^2)}{1-\cos(3x)}$.   (b) (5%) $\lim_{x\to 0}(\cos x)^{\csc(x^2)}$.   (c) (5%) $\lim_{x\to\infty} x^3\left(\frac{1}{x} - \sin(\frac{1}{x})\right)$.

**Solution:**

(a) Formally:

$$\lim_{x\to 0} \tan^{-1}(x^2) = \tan^{-1}(0) = 0$$

$$\lim_{x\to 0} 1 - \cos(3x) = 1 - \cos(0) = 0$$

Indeterminate form: $\frac{0}{0}$

L'Hospital's rule

$$\lim_{x\to 0} \frac{\tan^{-1}(x^2)}{1-\cos(3x)} = \lim_{x\to 0} \frac{\frac{1}{1+(x^2)^2}\cdot(2x)}{0-(-\sin(3x))\cdot 3} = \lim_{x\to 0} \frac{2x}{3(1+x^4)\sin(3x)}$$

From here either

$$\lim_{x\to 0} \frac{2x}{3(1+x^4)\sin(3x)} = \lim_{x\to 0} \frac{2}{9(1+x^4)}\cdot\frac{3x}{\sin(3x)} = \frac{2}{9}$$

or l'Hospital's rule again

$$\lim_{x\to 0} \frac{2x}{3(1+x^4)\sin(3x)} = \lim_{x\to 0} \frac{2}{12x^3\sin(3x)+9(1+x^4)\cos(3x)} = \frac{2}{9}$$

□

Short solution:

$$\lim_{x\to 0} \frac{\tan^{-1}(x^2)}{1-\cos(3x)} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to 0} \frac{2x}{3(1+x^4)\sin(3x)} = \lim_{x\to 0} \frac{2}{9(1+x^4)}\cdot\frac{3x}{\sin(3x)} = \frac{2}{9}$$

□

Grading:
- Correct indeterminate form (1%), they can manipulate the function first to get a different answer (but still correct)
- Show knowledge of l'Hospital's rule (1%) they get this point as long as they put the function into a fraction, consider whether it is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and start taking derivatives
- (3%) if they identify the indeterminate form and use l'Hospital's rule correctly once. Finishing the problem from there is worth (2%)
- Clearly showing that they read the problem wrong, but work and answer is correct (at most 3%)
- Minor simplification mistakes (-1%)
- Using l'Hospital's rule when it doesn't apply (-4%)

(b) Formally:

$$\lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} x^2 = 0^+$$

$$\lim_{x\to 0} \csc(x^2) = \infty$$

Indeterminate form: $1^\infty$

Use natural logarithm to obtain indeterminate forms for l'Hospital's rule

$$\lim_{x\to 0} (\cos x)^{\csc(x^2)} = \lim_{x\to 0} e^{\csc(x^2)\ln(\cos x)} = \lim_{x\to 0} e^{\frac{\ln(\cos x)}{\sin(x^2)}}$$

$$\lim_{x\to 0} \ln(\cos x) = 0$$

$$\lim_{x\to 0} \sin(x^2) = 0$$

L'Hospital's rule

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin(x^2)} = \lim_{x\to 0}\frac{\dfrac{1}{\cos x}\cdot(-\sin x)}{\cos(x^2)\cdot(2x)} = \lim_{x\to 0}\frac{-\sin x}{2x\cos x\cos(x^2)}$$

Either l'Hospital's rule again or use $\lim_{x\to 0}\frac{\sin x}{x} = 1$ to get

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin(x^2)} = -\frac{1}{2}$$

and

$$\lim_{x\to 0}(\cos x)^{\csc(x^2)} = e^{-1/2}$$

□

Short solution:

$$\lim_{x\to 0}(\cos x)^{\csc(x^2)}\overset{1^\infty}{=} e^{\lim_{x\to 0}\frac{\ln(\cos x)}{\sin(x^2)}}\overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} e^{\lim_{x\to 0}\frac{-\sin x}{2x\cos x\cos(x^2)}} = e^{-1/2}$$

□

Grading:
- Correct indeterminate form (1%), they can manipulate the function first to get a different answer (but still correct)
- Show knowledge of l'Hospital's rule (1%) they get this point as long as they put the function into a fraction, consider whether it is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and start taking derivatives
- (3%) if they identify the indeterminate form and use l'Hospital's rule correctly once. Finishing the problem from there is worth (2%)
- Clearly showing that they read the problem wrong, but work and answer is correct (at most 3%)
- Minor simplification mistakes (-1%)
- Using l'Hospital's rule when it doesn't apply (-4%)

(c) Formally:

$$\lim_{x\to\infty} x^3 = \infty$$

$$\lim_{x\to\infty}\frac{1}{x} - \sin\left(\frac{1}{x}\right) = 0$$

Indeterminate form: $\infty \cdot 0$

Direct method:

$$\lim_{x\to\infty} x^3\left(\frac{1}{x} - \sin\left(\frac{1}{x}\right)\right) = \lim_{x\to\infty}\frac{\frac{1}{x} - \sin\left(\frac{1}{x}\right)}{x^{-3}}$$

L'Hospital's rule

$$\lim_{x\to\infty}\frac{\frac{1}{x} - \sin\left(\frac{1}{x}\right)}{x^{-3}} = \lim_{x\to\infty}\frac{-x^{-2} - \cos\left(\frac{1}{x}\right)\cdot(-x^{-2})}{-3x^{-4}} = \lim_{x\to\infty}\frac{1-\cos\left(\frac{1}{x}\right)}{3x^{-2}}$$

L'Hospital's rule

$$\lim_{x\to\infty}\frac{1-\cos\left(\frac{1}{x}\right)}{3x^{-2}} = \lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)\cdot(-x^{-2})}{-6x^{-3}} = \lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{6x^{-1}}$$

L'Hospital's rule

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{6x^{-1}} = \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\cdot(-x^{-2})}{-6x^{-2}} = \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)}{6} = \frac{1}{6}$$

□

Easier method: Let $y = x^{-1}$,

$$\lim_{x\to\infty} x^3\left(\frac{1}{x} - \sin\left(\frac{1}{x}\right)\right) = \lim_{y\to 0^+}\frac{y-\sin y}{y^3}$$

L'Hospital's rule 2 times

$$\lim_{y\to 0^+}\frac{y-\sin y}{y^3} = \lim_{y\to 0^+}\frac{1-\cos y}{3y^2} = \lim_{y\to 0^+}\frac{\sin y}{6y} = \frac{1}{6}$$

□

Short solution:

$$\lim_{x\to\infty} x^3\left(\frac{1}{x} - \sin\left(\frac{1}{x}\right)\right)\overset{\infty\cdot 0}{=} \lim_{y\to 0^+}\frac{y-\sin y}{y^3}\underset{y=\frac{1}{x}}{}$$

$$\lim_{y\to 0^+} \frac{y - \sin y}{y^3} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{y\to 0^+} \frac{1-\cos y}{3y^2} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{y\to 0^+} \frac{\sin y}{6y} = \frac{1}{6}$$

□

Grading:

- Correct indeterminate form (1%), they can manipulate the function first to get a different answer (but still correct)
- Show knowledge of l'Hospital's rule (1%) they get this point as long as they put the function into a fraction, consider whether it is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and start taking derivatives
- (3%) if they identify the indeterminate form and use l'Hospital's rule correctly once. Finishing the problem from there is worth (2%)
- Clearly showing that they read the problem wrong, but work and answer is correct (at most 3%)
- Minor simplification mistakes (-1%)
- Using l'Hospital's rule when it doesn't apply (-4%)

6. (12%) A manufacturer determines that in order to sell $x$ units of product, the price per unit is given by the law

$$p(x) = 18000 - 0.5x^2 \quad \text{(in dollars)}.$$

On other hand, the cost of manufacturing $x$ units is determined by

$$C(x) = 1000 + 3000x \quad \text{(in dollars)},$$

where $C(0) = 1000$ is the fixed cost.

(a) (2%) Find the interval of $x$, denoted by $I$, on which the price $p(x)$ is non-negative. Note that we require $x \geq 0$.

(b) (2%) Let $T(x)$ be the profit function (Revenue = Units × Price, Profit = Revenue − Cost). Please find $T(x)$.

(c) (6%) Find the price per unit such that the profit $T$ is maximized on the interval $I$.

(d) (2%) Let $x_{\max}$ be the unit where the profit $T(x_{\max})$ is the maximum value. Will $x_{\max}$ change if we use a different fixed cost $C(0)$? Please explain your reasoning.

**Solution:**

(a) (2%) Requires $x \geq 0$ and $p(x) \geq 0$, that is,

$$0 \leq x \leq \sqrt{\frac{18000}{0.5}} \ (\approx 189.73).$$

Hence

$$I = \left[0, \sqrt{36000}\right] \text{(full credit for the correct answer. No partial credit).}$$

(b) (2%)

$$\begin{aligned} T(x) =& xp(x) - C(x) = x(18000 - 0.5x^2) - (1000 + 3000x) \\ =& -0.5x^3 + 15000x - 1000. \end{aligned}$$

Full credit for the correct answer. No partial credit. It is OK if students do not simplify $T(x)$.

(c) We first look for the critical points inside of $I$. That is, find $x$ such that

$$0 = T'(x) = -1.5x^2 + 15000,$$

i.e., $x_{\max} = 100$. (2%)

Check that $T$ attains the maximum at $x_{\max}$. Note that

$$\begin{aligned} &T(0) = -1000, \quad T(\sqrt{36000}) = -1000 - 3000 \times \sqrt{36000} < 0, \\ &T(x_{\max}) = 100(18000 - 0.5 \times 100^2) - (1000 + 3000 \times 100) = 999000 > 0. \end{aligned}$$

Hence $T$ attains the maximum at $x_{\max}$. (2%). Students do not need to compute the exact values of $T$ at particular points. Determining the signs of $T$ at those points is sufficient.

At $x_{\max} = 100$, $p(x_{\max}) = p(100) = 18000 - 0.5 \times 100^2 = 13000$. (2%)

(d) (2%) $x_{\max}$ will not change if we use a different fixed value $C(0)$ since $T'(x)$ does not depend on $C(0)$.

7. (16%) Let

$$f(x) = \ln\left|\frac{2x+1}{x-1}\right|.$$

(a) (1%) Write down the domain of $f(x)$.

(b) (4%) Compute $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $f(x)$.

(c) (4%) Compute $f''(x)$. Write down the interval(s) on which $f(x)$ is concave upward and the interval(s) on which $f(x)$ is concave downward.

(d) (4%) Find all vertical asymptotes and horizontal asymptotes of $y = f(x)$.

(e) (3%) Sketch the graph of $y = f(x)$.

**Solution:**

(a) The domain of $f(x)$ is $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$.

(b)

$$f'(x) = \frac{x-1}{2x+1}\,\frac{2(x-1)-(2x+1)}{(x-1)^2} = \frac{-3}{(2x+1)(x-1)} = -3\frac{1}{2x^2-x-1}. \quad (2\%)$$

$f(x)$ is increasing on $(-\frac{1}{2}, 1)$ (1%) and decreasing on $(-\infty, -\frac{1}{2}) \cup (1, \infty)$ (1%).

(c)

$$f''(x) = -3\left(\frac{1}{2x^2-x-1}\right)' = 3\frac{4x-1}{(2x^2-x-1)^2}. \quad (2\%)$$

$f(x)$ is concave upward on $(\frac{1}{4}, \infty)$ (1%) and concave downward on $(-\infty, \frac{1}{4})$ (1%).

(d)

$$\lim_{x\to 1} f(x) = \infty, \quad \lim_{x\to -\frac{1}{2}} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \ln 2.$$

Hence the lines $x = 1$, $x = -\frac{1}{2}$ are vertical asymptotes of $y = f(x)$ (2%), and the line $y = \ln 2$ is a horizontal asymptote of $y = f(x)$ (2%).

(e)