1. (18%) (a) (4%) Compute $\lim_{x\to\infty} \frac{[x]}{x}$, where $[\cdot]$ is the greatest integer function.

(b) (4%) Compute $\lim_{x\to 0^-} \frac{x}{\sqrt{1-\cos kx}}$, where $k>0$ is a constant.

(c) (4%) Compute $\lim_{x\to 0}(\csc x - \frac{1}{e^x-1})$.

(d) (6%) Find constants $a, b \in \mathbb{R}$ such that $a \neq 0$, $b>0$ and $\lim_{x\to 0^+}(\cos x)^{a/x^b} = 3$.

Solution:

(a) Since $\underbrace{x-1<[x]\le x}_{(3\%)}$, by Squeeze Theorem, the limit equals to $\underbrace{1}_{(1\%)}$.

(b)

$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos kx}} = \underbrace{\lim_{x\to 0^-}\frac{x\sqrt{1+\cos(kx)}}{|\sin(kx)|}}_{(2\%)} = \underbrace{\sqrt{2}}_{(1\%)}\cdot\lim_{x\to 0^-}\frac{x}{|\sin(kx)|} = \sqrt{2}\cdot\lim_{x\to 0^-}\frac{x}{-\sin kx} = \underbrace{-\frac{\sqrt{2}}{k}}_{(1\%)}$$

Remark. -1% for omitting the absolute value.

(c)

$$\begin{aligned}\lim_{x\to 0}(\csc x - \frac{1}{e^x-1}) &= \lim_{x\to 0}\frac{e^x-1-\sin x}{(\sin x)(e^x-1)}\\ &\overset{[\frac{0}{0}],\text{L'H}}{=} \lim_{x\to 0}\frac{e^x-\cos x}{(e^x-1)\cos x + e^x\sin x} \quad (2\%)\\ &\overset{[\frac{0}{0}],\text{L'H}}{=} \lim_{x\to 0}\frac{e^x+\sin x}{-(e^x-1)\sin x + 2e^x\cos x + e^x\sin x}\\ &= \frac{1}{2} \quad (2\%)\end{aligned}$$

(d) We need to solve $\lim_{x\to 0^+}\frac{a\ln(\cos x)}{x^b} = \ln 3$. (2%)
It is a $\frac{0}{0}$ form since $b>0$. Apply L'Hospital's Rule to study

$$\lim_{x\to 0^+}\frac{a}{b}\frac{-\sin x}{(\cos x)x^{b-1}} = -\frac{a}{b}\lim_{x\to 0^+}\frac{\sin x}{x^{b-1}} = \ln 3 \quad (2\%)$$

Clearly $\underbrace{b=2}_{(1\%)}$, $a=-b\ln 3 = \underbrace{-2\ln 3}_{(1\%)}$.

2. (8%) Compute the following derivatives.

   (a) (4%) $\dfrac{d}{dx}\left(2^{2^x} + x^{x^2}\right)$.

   (b) (4%) $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x+a}{x-a}}\right)$, where $a \neq 0$ is a constant.

Solution:

(a) By the chain rule

$$\frac{d}{dx}\left(2^{2^x}\right) = (\ln 2)^2 2^x \cdot 2^{2^x} \quad \text{(2 pts)}$$

Let $f(x) = x^{x^2}$. $\ln|f(x)| = x^2 \cdot \ln|x| \ldots (*)$

$\frac{d}{dx}(*) \Rightarrow \frac{f'(x)}{f(x)} = 2x\ln|x| + x$

(1 pt for trying to do logarithmic differentiation. e.g. compute $\ln|f(x)|$, and know that $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$.)

$\Rightarrow f'(x) = x^{x^2}(2x\ln|x| + x)$ (1 pt)

(It is O.K. to write $\ln x$ instead of $\ln|x|$. Because the domain of $f(x)$ is $\{x|x > 0\}$.)

(b) $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{a}{x^2+a^2}$

(2 pts.

Wrong ans: $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{1+\left(\frac{x}{a}\right)^2} = \frac{a^2}{x^2+a^2} \Rightarrow$ 1 pt)

Compute $\frac{d}{dx}\ln\sqrt{\frac{x+a}{x-a}}$.

sol 1: $\ln\sqrt{\frac{x+a}{x-a}}$ is defined for $|x| > |a|$.

For $|x| > |a|$, $\ln\sqrt{\frac{x+a}{x-a}} = \frac{1}{2}(\ln|x+a| - \ln|x-a|)$

$\frac{d}{dx}\left(\ln\sqrt{\frac{x+a}{x-a}}\right) = \frac{1}{2}\frac{d}{dx}(\ln|x+a| - \ln|x-a|) = \frac{1}{2}\left(\frac{1}{x+a} - \frac{1}{x-a}\right) = \frac{-a}{x^2-a^2}$ (2 pts)

sol 2:

$\frac{d}{dx}\left(\ln\sqrt{\frac{x+a}{x-a}}\right) = \frac{1}{\sqrt{\frac{x+a}{x-a}}}\frac{d}{dx}\left(\sqrt{\frac{x+a}{x-a}}\right) = \frac{1}{2}\frac{x-a}{x+a}\frac{-2a}{(x-a)^2} = \frac{-a}{x^2-a^2}$ (2 pts)

Hence $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x+a}{x-a}}\right) = a\left[\frac{1}{x^2+a^2} - \frac{1}{x^2-a^2}\right] = \frac{-2a^3}{x^4-a^4}$

3. (10%) (a) (6%) Suppose that $f(x) \le g(x) \le h(x)$ and $f(x), h(x)$ are differentiable at a with $f(a) = h(a)$, $f'(a) = h'(a)$. Show that $g(x)$ is differentiable at a and find $g'(a)$.

(b) (4%) Give an example of functions $f(x)$, $g(x)$, and $h(x)$ such that $f(x) \le g(x) \le h(x)$, $f'(a) = h'(a)$ but $g(x)$ is not differentiable at a.

Solution:

(a) (+1) First we show that $g(a) = f(a) = h(a)$. Since $f(x) \le g(x) \le h(x)$ with $f(a) = h(a)$, we have $g(a) = f(a)$.

(Use squeeze lemma, but do not carefully distinguish the sign: (+3))

We compute $\lim_{x\to a^+} \frac{g(x)-g(a)}{x-a}$. For $x > a$, we have

$$\frac{f(x) - f(a)}{x - a} \le \frac{g(x) - g(a)}{x - a} \le \frac{h(x) - h(a)}{x - a}$$

followed by $f(x) \le g(x) \le h(x), f(a) = g(a) = h(a)$ and $x - a > 0$. Since the $\lim_{x\to a^+}$ of the left and right terms in the above inequalities exist and equal $f'(a) = h'(a)$, this forces that $\lim_{x\to a^+} \frac{g(x)-g(a)}{x-a}$ exists and equals $f'(a) = h'(a)$.

For $\lim_{x\to a^-} \frac{g(x)-g(a)}{x-a}$, one repeats the argument with the reversed inequalities

$$\frac{f(x) - f(a)}{x - a} \ge \frac{g(x) - g(a)}{x - a} \ge \frac{h(x) - h(a)}{x - a}.$$

We conclude that $g(x)$ is differentiable at a and $g'(a) = f'(a)$.

(b) For example,

$$f(x) = x - 1, \quad g(x) = [x], \quad h(x) = x$$

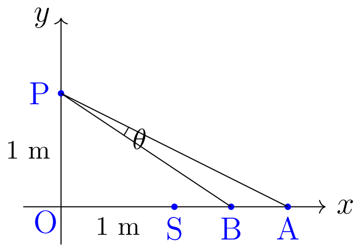
and a is any integer. Then $f'(x) = 1 = h'(x)$ but $g(x)$ is not differentiable at a since it is not continuous at a.

(Sketch a graph without explanation: (+2); sketch a graph with correct explanations: (+4).)

4. (12%) An observer stands at point P which is one meter from a straight path. Let O be the point on the path that is closest to P, and S be the point on the path that is one meter to the right of O. Two runners A and B start at S and run away from O along the path. Let θ be the observer's angle of sight between the runners.

(a) (6%) Suppose that when A and B start at S, $\frac{d\theta}{dt} = \frac{1}{4}$ rad/sec. Find the relative velocity between A and B at S.

(b) (6%) Suppose that A runs twice as fast as B. Find the maximum value of θ.



Solution:

(a) Suppose that at time t(sec), A is $A(t)$ meters to the right of O and B is $B(t)$ meters to the right of O. Then $A(0) = B(0) = 1$.
$\theta(t) = \tan^{-1} A(t) - \tan^{-1} B(t)$.
(3 pts for assigning notations and the correct equation.)
$\frac{d\theta}{dt} = \frac{A'(t)}{1+(A(t))^2} - \frac{B'(t)}{1+(B(t))^2}$
(2 pts for differentiation)
At $t = 0$, $\frac{d\theta}{dt} = \frac{1}{4} = \frac{A'(0)}{1+(A(0))^2} - \frac{B'(0)}{1+(B(0))^2} = \frac{1}{2}(A'(0) - B'(0))$
i.e. $A'(0) = B'(0) = \frac{1}{2}$ m/s. (1 pt for plugging in $t = 0$)
Ans: The relative velocity between A and B at point S is $\frac{1}{2}$ m/s.

(b) Sol 1:
When B is x meters to the right of S, A is $2x$ meters to the right of S.
$\theta(x) = \tan^{-1}(2x + 1) - \tan^{-1}(x + 1)$ for $x > 0$.
(2 pts for assigning notations and deriving the correct equation with correct domain.)
$\frac{d\theta}{dx} = \frac{2}{1+(2x+1)^2} = \frac{1}{1+(x+1)^2} = \frac{-2x^2+2}{(4x^2+4x+2)(x^2+2x+2)}$,
$\frac{d\theta}{dx} = 0 \Rightarrow x = \pm 1$
(2 pts for computing $\frac{d\theta}{dx}$. 1 pt for solving $\frac{d\theta}{dx} = 0$.)
$\frac{d\theta}{dx} > 0$ for $0 < x < 1$ and $\frac{d\theta}{dx} < 0$ for $x > 1$.
Hence θ obtains the absolute maximum when $x = 1$ i.e. when B is 1 meter and A is 2 meters to the right of S.
(1 pt for explaining that the critical number is the absolute maximum.)

Sol 2:
Suppose that the velocity of B is v m/s and the velocity of A is 2v m/s.
Then offer t seconds B is $1 + vt$ meters to the right of O and A is $1 + 2vt$ meters to the right of O.
$\theta(t) = \tan^{-1}(1 + 2vt) - \tan^{-1}(1 + vt)$ for $t > 0$.
(2 pts for assigning notations and deriving the correct equation with correct domain.)
$\frac{d\theta}{dt} = \frac{2v}{1+(1+2vt)^2} - \frac{v}{1+(1+vt)^2} = v\left[\frac{-2v^2t^2+2}{(1+(1+2vt))^2(1+(1+vt))^2}\right]$ (2 pts)
$\frac{d\theta}{dt} = 0 \Rightarrow vt = \pm 1$ (1 pt)
$\frac{d\theta}{dt} > 0$ for $0 < vt < 1$, $\frac{d\theta}{dt} < 0$ for $vt > 1$.
Hence θ obtains the absolute maximum when $vt = 1$ i.e. when B is 2 meters to the right of O and A is 3 meters to the right of O. (1 pt)

5. (14%) Consider the equation $y^5 + 1.009y^3 + y = 3$.

(a) (6%) Show that the equation has exactly one real solution.

(b) (4%) Given $y^5 + xy^3 + y = 3$, find $\frac{dy}{dx}$ at $(1, 1)$.

(c) (4%) Use a linear approximation to estimate the real root of $y^5 + 1.009y^3 + y = 3$.

Solution:

(a) Let $g(y) = y^5 + 1.009y^3 + y - 3$. Since

$$\lim_{y\to\infty} g(y) = \infty, \quad \lim_{y\to-\infty} g(y) = -\infty,$$

the Intermediate Value Theorem implies that g has real roots. **(3 points)**

Moreover, since

$$g'(y) = 5y^4 + 3.027y^2 + 1 > 0,$$

g is strictly increasing. In particular, g has at most one real root. **(3 points)**

(b) By the argument in (a), y is implicitly defined as a function of x via the equation

$$y^5 + xy^3 + y = 3$$

near $(x, y) = (1, 1)$. Differentiating both sides of the above equation with respect to x gives

$$\left(5y^4 + 3xy^2 + 1\right)\frac{dy}{dx} + y^3 = 0 \quad \text{(3 points)}.$$

Substituting $(x, y) = (1, 1)$ into the above equation gives

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = -\frac{1}{9}. \quad \text{(1 point)}$$

(c) Let's denote $y = f(x)$. Note that $f(1) = 1$ and $f'(1) = -\frac{1}{9}$. **(2 points)** Then

$$f(1.009) \approx f(1) + f'(1) * 0.009 = 1 - \frac{1}{9} * 0.009 = 0.999 \quad \text{(2 points)}.$$

6. (18%) Suppose that f is differentiable and one-to-one on $(-1,1)$, $f'(x) = 1 + f^2(x)$, and $\lim_{x\to 0}\frac{f(x)}{x}$ exists.

(a) (4%) Find $f(0)$ and $\lim_{x\to 0}\frac{f(x)}{x}$.

(b) (4%) Show that $f(x)$ is increasing on $(-1,1)$ and determine the concavity of $y = f(x)$ on $(-1,1)$.

(c) (6%) Prove that $f(x) \geq x$ for $x \in (0,1)$. Then prove that $f(x) \geq x + \frac{x^3}{3}$ for $x \in (0,1)$.

(d) (2%) Find $\frac{d}{dx}(f^{-1}(x))$.

(e) (2%) Find $f^{-1}(x)$ and $f(x)$.

Solution:

(a) Let $L = \lim_{x\to 0}\frac{f(x)}{x}$. Since f is differentiable, it is continuous. Therefore,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{f(x)}{x}\cdot x = L\cdot 0 = 0,$$

and hence

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1 + f(0)^2 = 1.$$

- 求出$f(0) = 0$可得1分。
- 有提到f連續性或考慮$\lim_{x\to 0} f(x)$可得1分。
- 求出$\lim_{x\to 0}\frac{f(x)}{x} = 1$可得1分。
- 寫出得到$\lim_{x\to 0}\frac{f(x)}{x} = 1$的理由（如運用導數定義或正確使用L'Hôpital法則）可得1分。

(b) Since $f'(x) = 1 + f(x)^2 \geqslant 1 > 0$, f is (strictly) increasing. To determine the concavity of f, we compute the 2nd derivative of f:

$$f'' = (f')' = (1+f^2)' = 2ff'.$$

$f''(x)$ and $f(x)$ have the same sign for every $x \in (-1,1)$ since $f'(x) > 0$. We have obtained in (a) that $f(0) = 0$. Therefore $f(x) > 0$ resp. < 0 if $x \in (0,1)$ resp. $(-1,0)$, as shows that f is concave upward resp. downward on $(0,1)$ resp. $(-1,0)$.

- 看出$f' > 0$並由此導出$F \nearrow\!\nearrow$可得2分。
- 利用$f' = 1 + f^2$來計算f''並企圖以此判斷凹向來可得1分。
- f''的計算正確且判斷凹向的理由亦正確可再得1分。

(c) Consider the function $h(x) = f(x) - x$. By (a) and (b) we have

$$h(0) = f(0) = 0 = 0 \quad\text{and}\quad h'(x) = f'(x) - 1 = f(x)^2 > 0,$$

and hence h is increasing on $[0,1)$ and $f(x) - x = h(x) > h(0) = 0$ for $x \in (0,1)$. Now consider $g(x) = f(x) - x - \frac{x^3}{3}$. We have

$$g(0) = f(0) - 0 = 0 \quad\text{and}\quad g'(x) = f'(x) - 1 - x^2 = f(x)^2 - x^2 > 0\ (x \in (0,1)),$$

where the last inequality holds by the first part of (c), as we just obtained. Therefore, $f(x) - x - \frac{x^3}{3} = g(x) > g(0) = 0$ for every $x \in (0,1)$.

- 原則上第一、二部分各佔3分。

(d) We have

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)} = \frac{1}{1 + f\big(f^{-1}(x)\big)^2} = \frac{1}{1+x^2}.$$

- 列出$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$可得1分。
- 計算正確1分。

(e) From the result of (d) we see that f^{-1} is a function whose derivative is $\frac{1}{1+x^2}$. We know one such function, namely, $\tan^{-1} x$, and hence the derivative of $f^{-1}(x) - \tan^{-1} x$ is 0. A function on an interval with derivative 0 everywhere is a constant function by the mean value theorem. Therefore, $f^{-1}(x) = \tan x + C$ for some constant C. Finally, $C = f^{-1}(0) - \tan^{-1} 0 = 0$, and hence $f^{-1}(x) = \tan^{-1} x$ and $f(x) = \tan x$.

- 提到$f^{-1}(x) = \tan x$可得1分。
- 有考慮常數C如何決定者可再得1分。

7. (20%) Let $f(x) = x(\ln|x|)^2$.

(a) (2%) Find the domain of $f(x)$. Is f an odd function or even function?

(b) (2%) Compute $\lim_{x\to 0} f(x)$.

(c) (4%) Compute $f'(x)$. Find the interval(s) of increase and interval(s) of decrease of $f(x)$.

(d) (2%) Find local maximum and local minimum values of $f(x)$.

(e) (4%) Compute $f''(x)$. Find the interval(s) on which $f(x)$ is concave upward. Find the interval(s) on which $f(x)$ is concave downward.

(f) (2%) Find the point(s) of inflection of $y = f(x)$.

(g) (1%) Find the asymptote(s) (vertical, horizontal, or slant) of $y = f(x)$.

(h) (3%) Sketch the graph of $f(x)$.

Solution:

(a) f is defined on $\mathbb{R} \setminus \{0\}$. **(1 point)**

Since $f(-x) = -f(x)$, f is an odd function. **(1 point)**

(b)

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \left(\frac{\ln x}{\frac{1}{\sqrt{x}}}\right)^2 = \lim_{x\to 0+} 4\left(\frac{\ln \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}\right)^2 = \lim_{y\to\infty} 4\left(\frac{\ln y}{y}\right)^2 = 0. \quad \text{(2 points)}$$

Since f is odd, we have

$$\lim_{x\to 0-} f(x) = -\lim_{x\to 0+} f(x) = 0.$$

Thus, $\lim_{x\to 0} f(x) = 0$.

(c)

$$f'(x) = (\ln|x|)^2 + 2\ln|x| = \ln|x|(\ln|x| + 2). \quad \text{(2 points)}$$

So f is increasing on $(-\infty, -1) \cup (-e^{-2}, 0) \cup (0, e^{-2}) \cup (1, \infty)$ and decreasing on $(-1, -e^{-2}) \cup (e^{-2}, 1)$. **(2 points)**

(d) By the First Derivative Test, f has local maxima

$$f(-1) = 0, \quad f\left(e^{-2}\right) = 4e^{-2} \quad \text{(1 point)}$$

and local minima

$$f\left(-e^{-2}\right) = -4e^{-2}, \quad f(1) = 0. \quad \text{(1 point)}$$

(e)

$$f''(x) = \frac{2}{x}\ln|x| + \frac{2}{x} = \frac{2}{x}(\ln|x| + 1). \quad \text{(2 points)}$$

So f is concave upward on $(-e^{-1}, 0) \cup (e^{-1}, \infty)$ and concave downward on $(-\infty, -e^{-1}) \cup (0, e^{-1})$. **(2 points)**

(f) The inflection points of the graph of f are $(-e^{-1}, -e^{-1})$, $(0, 0)$, and (e^{-1}, e^{-1}). **(2 points)**

(g) Obviously there is no vertical asymptotes. Since

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} (\ln|x|)^2 = \infty,$$

f has no horizontal or slant asymptotes. **(1 point)**

(h) **(3 points)**