1082 模03-07班期考手寫題詳解及評分標準

1. (20%) As a financially independent NTU student, you spend 8 hours at school and need 8 hours of sleep every day. This leaves you 8 hours of time to allocate between studying t_1 hours at the K-book center, singing for t_2 hours at the KTV, and teaching t_3 hours hours as a private tutor for high school kids. Suppose that the price of staying at the K-book center is p_1 dollars per hour, the price of singing at the KTV is p_2 dollars per hour, and your hourly wage as a private tutor is w dollars. Then, your consumer problem has both a time constraint $t_1 + t_2 + t_3 = 8$ and a budget constraint $p_1t_1 + p_2t_2 = wt_3$. Suppose that your utility function is $f(t_1, t_2, t_3) = \frac{1}{2}\ln t_1 + \frac{1}{2}\ln t_2$.
   (a) (14%) Find t_1^*, t_2^* and t_3^* that maximizes the utility $f(t_1, t_2, t_3)$. Find the maximum utility $f(t_1^*, t_2^*, t_3^*)$. Write your answers in terms of p_1, p_2, and w.
   (b) (6%) If your hourly wage w increases, find the rate of change for the maximum utility $f(t_1^*, t_2^*, t_3^*)$ with respect to w. i.e. compute $\frac{\partial}{\partial w}f(t_1^*, t_2^*, t_3^*)$. Write your answer in terms of p_1, p_2, and w.

Solution:

(a) **Solution 1:**

Let $g(t_1, t_2, t_3) = t_1 + t_2 + t_3$, $h(t_1, t_2, t_3) = p_1t_1 + p_2t_2 - wt_3$

We want to find the maximum value of $f(t_1, t_2, t_3) = \frac{1}{2}\ln t_1 + \frac{1}{2}\ln t_2$ under constraints $g = 8$, $h = 0$.

By the method of Lagrange multipliers, we solve the system of equations

$$\begin{cases} f_{t_1} = \lambda g_{t_1} + \mu h_{t_1} \\ f_{t_2} = \lambda g_{t_2} + \mu h_{t_2} \\ f_{t_3} = \lambda g_{t_3} + \mu h_{t_3} \\ g = 8 \\ h = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2}\frac{1}{t_1} = \lambda + p_1\mu \cdots\cdots ① \\ \frac{1}{2}\frac{1}{t_2} = \lambda + p_2\mu \cdots\cdots ② \\ 0 = \lambda - \mu w \cdots\cdots ③ \\ t_1 + t_2 + t_3 = 8 \cdots\cdots ④ \\ p_1t_1 + p_2t_2 - wt_3 = 0 \cdots\cdots ⑤ \end{cases}$$

(5pts for correct setting and equations.)

$$\begin{cases} ③ \Rightarrow \lambda = \mu \cdot w \\ ① \Rightarrow t_1 = \frac{1}{2\mu(w+p_1)} \\ ② \Rightarrow t_2 = \frac{1}{2\mu(w+p_2)} \\ ④ \times w + ⑤ \Rightarrow (w+p_1)t_1 + (w+p_2)t_2 = 8w \Rightarrow \frac{1}{u} = 8w. \text{Hence } t_1 = \frac{4w}{w+p_1}, t_2 = \frac{4w}{w+p_2} \\ ⑤ \Rightarrow t_3 = \frac{4p_1}{w+p_1} + \frac{4p_2}{w+p_2} \end{cases}$$

(7 pts for solving equations)

Hence we have only one solution $(t_1^*, t_2^*, t_3^*) = \left(\frac{4w}{w+p_1}, \frac{4w}{w+p_2}, \frac{4p_1}{w+p_1} + \frac{4p_2}{w+p_2}\right)$ with $\mu = \frac{1}{8w}$, $\lambda = \frac{1}{8}$.

We know that f should obtain maximum value on the constraint set. Therefore $f(t_1^*, t_2^*, t_3^*)$ is the maximum utility $f(t_1^*, t_2^*, t_3^*) = \frac{1}{2}\ln\left(\frac{4w}{w+p_1}\right) + \frac{1}{2}\ln\left(\frac{4w}{w+p_2}\right)$.

(2 pts for correct answers)

Solution 2:

Solve $\begin{cases} t_1 + t_2 + t_3 = 8 \\ p_1t_1 + p_2t_2 = wt_3 \end{cases} \Rightarrow \begin{cases} (p_2 - p_1)t_1 = 8p_2 - (w+p_2)t_3 \\ (P_2 - P_1)t_2 = -8p_1 + (w+p_1)t_3 \end{cases}$

$\Rightarrow t_1 = \frac{8p_2}{p_2-p_1} - \frac{w+p_2}{p_2-p_1}t_3$, $t_2 = \frac{-8p_1}{p_2-p_1} + \frac{w+p_1}{p_2-p_1}t_3$.

$$f(t_1, t_2, t_3) = \frac{1}{2}\ln t_1 + \frac{1}{2}\ln t_2 = \frac{1}{2}\ln\left(\frac{8p_2}{p_2-p_1} - \frac{w+p_2}{p_2-p_1}t_3\right) + \frac{1}{2}\ln\left(\frac{-8p_1}{p_2-p_1} + \frac{w+p_1}{p_2-p_1}t_3\right)$$

is a function of single variable t_3.

Find t_3 such that the above function is maximized. 6pts

Solve the correct answer 8pts.

(b) $\frac{\partial}{\partial w}f(t_1^*, t_2^*, t_3^*)\frac{1}{2}\left(\frac{1}{w} - \frac{1}{w+p_1}\right) + \frac{1}{2}\left(\frac{1}{w} - \frac{1}{w+p_2}\right) = \frac{1}{w} - \frac{1}{2}\left(\frac{1}{w+p_1}\right) - \frac{1}{2}\left(\frac{1}{w+p_2}\right)$...... 6pts

2. (16%) We define the improper integral over the entire plane $\mathbb{R}^2$

$$I = \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A \quad \text{as} \quad \lim_{a\to\infty} \iint_{D_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A,$$

where D_a is the disk with radius a and center the origin.

(a) (10%) Find $\iint_{D_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A$. Compute $\lim_{a\to\infty} \iint_{D_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A$

(b) (6%) The integral I can be also defined as $\lim_{a\to\infty} \iint_{S_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A$, where $S_a = [-a,a]\times[-a,a]$. From (a), Compute $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\mathrm{d}x$.

Solution:

(a)

$$\begin{aligned}
\iint_{D_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A &= \int_0^{2\pi}\int_0^a e^{-\frac{1}{2}r^2} r\mathrm{d}r\mathrm{d}\theta \quad \text{(2 points for writing down this)} \\
&= \int_0^{2\pi}\int_{r=0}^{r=a} e^{-\frac{1}{2}r^2}\mathrm{d}\left(\frac{1}{2}r^2\right)\mathrm{d}\theta \quad \text{(2 points for this change of variables)} \\
&= 2\pi(-1)e^{-\frac{1}{2}r^2}\Big|_{r=0}^{r=a} \\
&= -2\pi e^{-\frac{1}{2}a^2} + 2\pi \quad \text{(3 points for obtaining this)}
\end{aligned}$$

$\therefore \lim_{a\to\infty} \iint_{D_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A = \lim_{a\to\infty}\left(-2\pi e^{-\frac{1}{2}a^2} + 2\pi\right) = 2\pi$ (3 points)

(b)

$$\begin{aligned}
I &= \lim_{a\to\infty} \iint_{S_a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}A \\
&= \lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-\frac{1}{2}(x^2+y^2)}\mathrm{d}x\mathrm{d}y \quad \text{(2 points)} \\
&= \lim_{a\to\infty} \int_{-a}^{a} e^{-\frac{1}{2}x^2}\mathrm{d}x \int_{-a}^{a} e^{-\frac{1}{2}y^2}\mathrm{d}y = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\mathrm{d}x \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\mathrm{d}y \quad \text{(2 points)}
\end{aligned}$$

$\therefore \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\mathrm{d}x = \sqrt{2\pi}$ (2 points)

3. (24%) Evaluate $\iint_R \sin\left(\frac{y-x}{y+x}\right) \mathrm{d}A$, where R is the trapezoidal region with vertices $(1,1)$, $(2,2)$,$(0,2)$, and $(0,4)$ by answering the following questions:

(a) (6%) Let $u = y - x$, $v = y + x$. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

(b) (6%) Plot the region S in the uv-plane corresponding to R and label all the vertices of it.

(c) (12%) Evaluate $\iint_R \sin\left(\frac{y-x}{y+x}\right) \mathrm{d}A$.

Solution:

(a) $u = y - x$, $v = y + x \implies x = \frac{1}{2}(v-u)$, $y = \frac{1}{2}(u+v)$ (2pts) $\implies \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{-1}{2}$. (4pts)

(b)
- (1pt) The equation of the line passing through $(1,1)$ and $(2,2)$ is $y - x = 0 \implies u = 0$.
- (1pt) The equation of the line passing through $(1,1)$ and $(0,2)$ is $y + x = 2 \implies v = 2$.
- (1pt) The equation of the line passing through $(0,2)$ and $(0,4)$ is $x = 0 \implies \frac{1}{2}(v-u) = 0 \implies u-v = 0$.
- (1pt) The equation of the line passing through $(0,4)$ and $(2,2)$ is $y + x = 4 \implies v = 4$

(2pts) The vertices of the trapezoidal region are $(0,2)$, $(2,2)$, $(4,4)$, and $(0,4)$.

(c) $\iint_R \sin\left(\frac{y-x}{y+x}\right) dA$

$= \int_2^4 \int_{u=0}^{u=v} \sin(u/v) \left|\frac{-1}{2}\right| du\, dv$ (3pts)

$= \int_2^4 -\frac{v}{2} \cos(u/v)\Big|_{u=0}^{u=v} dv$ (3pts)

$= \int_2^4 \frac{v}{2} (1 - \cos 1)\, dv$ (3pts)

$= 3\,(1 - \cos 1)$. (3pts)