1. (15%) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Please state the tests which you use.

   (a) (5%) $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n^2+4}-n)$.

   (b) (5%) $\sum_{n=1}^{\infty}\frac{\sin(n)}{n\sqrt{n}}$.

   (c) (5%) $\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$.

Solution:

(a) First of all

$$\sqrt{n^2+4}-n=\frac{4}{\sqrt{n^2+4}+n}. \quad (2 \text{ pts})$$

Since $\lim_{n\to\infty}\frac{4}{\sqrt{n^2+4}+n}=0$ and $\frac{4}{\sqrt{n^2+4}+n}$ is decreasing, then the series converges by Alternating Series Test. (+1 pts)

It is not absolutely convergent by using Limit Comparison with $\sum_{n=1}^{\infty}\frac{2}{n}$ (+2 pts)

(b) Since $|\sin(n)|\le 1$, we have

$$\left|\frac{\sin(n)}{n\sqrt{n}}\right|\le\frac{1}{n\sqrt{n}}. \quad (2 \text{ pts})$$

Note that $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ is convergent, (+1 pt) hence $\sum_{n=1}^{\infty}\frac{\sin(n)}{n\sqrt{n}}$ is absolutely convergent. (+ 2 pts)

(c) Let $a_n=\frac{3^n n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n}=3\left(\frac{n}{n+1}\right)^n,$$

(2 pts) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{3}{e}>1$. (3 pts)

Hence the series is divergent (+ 2 pts).

2. (14%) Find the interval of convergence of the power series and find the function represented by it.

(a) (7%) $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n(n-1)}$.

(b) (7%) $\sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}x^n$.

Solution:

(a) Let $a_n = (-1)^n \cdot \frac{x^n}{n(n-1)}$. By ration test, we have

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \Leftrightarrow \lim_{n\to\infty} \frac{(n-1)|x|}{n+1} < 1 \Leftrightarrow |x| < 1 \Leftrightarrow -1 < x < 1.$ (1 pt)

If $x = -1$, then $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges by limit comparison test. (1 pt)

If $x = 1$, then $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n(n-1)} = \sum_{n=2}^{\infty}(-1)^n \cdot \frac{1}{n(n-1)}$ converges since $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges, by (limit) comparison test. (1 pt)

Hence the interval of convergence is $-1 \le x \le 1$.

Let $S(x) = \sum_{n=2}^{\infty}(-1)^n \cdot \frac{x^n}{n(n-1)}$ for $-1 \le x \le 1$. Then

$$S'(x) = \sum_{n=2}^{\infty}(-1)^n \cdot \frac{x^{n-1}}{n-1}, \quad S'(0) = 0,$$

$$S''(x) = \sum_{n=2}^{\infty}(-1)^n \cdot x^{n-2} = \sum_{n=0}^{\infty}(-1)^n \cdot x^n = \frac{1}{1+x}. \qquad \text{(2 pts)}$$

So we have

$$S'(x) = \int_0^x S''(t)dt = \int_0^x \frac{1}{1+t}dt = \ln(1+x),$$

$$S(x) = \int_0^x S'(t)dt = \int_0^x \ln(1+t)dt = (x+1)\ln(x+1) - x. \qquad \text{(2 pts)}$$

Combining these results, we have $(x+1)\ln(x+1) - x = \sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n(n-1)}$ for $-1 \le x \le 1$.

P.S. If you mention and use result $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ then you can get 1 credit without using ratio test. But you need to check the endpoint ± 1 to get the other 2 credits.

(b) Let $a_n = \frac{n+1}{(n+2)!}x^n$. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)x^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{(n+1)x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)}{(n+1)(n+3)}\right||x| = 0 < 1 \qquad \text{(2 pts)}$$

for each real number x, by ratio test, the interval of convergence of $\sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}x^n$ is $\mathbb{R}$. (1 pt)

Consider $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}x^n$. Then we have

$$x^2 f(x) = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}x^{n+2}$$

$$\frac{d}{dx}[x^2 f(x)] = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} \cdot (n+2) \cdot x^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!}x^{n+1} = xe^x \qquad \text{(2 pts)}$$

$$x^2 f(x) = \int_0^x te^t dt = xe^x - e^x + 1$$

$$f(x) = \begin{cases} \frac{1}{x^2}(xe^x - e^x + 1) & ,\text{if } x \ne 0, \\ \frac{1}{2} & ,\text{if } x = 0. \end{cases} \qquad \text{(2 pts)}$$

P.S.

(a) We also accept the answer $f(x) = \frac{1}{x^2}(xe^x - e^x + 1)$.

(b) If you mention and use result $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$, then you can get all 3 credits of interval of convergence without using ratio test.

3. (15%)

(a) (4%) Find the Maclaurin series for $g(x) = x\int_0^x e^{t^2}\,dt$.

(b) (3%) Find $g^{(2020)}(0)$.

(c) (5%) Find the first three nonzero terms of the Maclaurin series for $f(x) = \begin{cases} \dfrac{\tan x}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$.

(d) (3%) Find $\displaystyle\lim_{x\to 0} \frac{x^2 f(x) - g(x)}{x^6}$.

Solution:

(a)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} t^{2n} = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots$$

$$g(x) = x\int_0^x e^{t^2}\,dt = x\sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+2} = x^2 + \frac{x^4}{3} + \frac{x^6}{5\cdot 2!} + \frac{x^8}{7\cdot 3!} + \cdots$$

(b)

$$g^{(2020)}(0) = c_{2020}\cdot (2020)!$$

Here c_{2020} is the coefficient of x^{2020} in the Maclaurin series for $g(x)$. Take $2n+2 = 2020,\ \ n = 1009$

$$g^{(2020)}(0) = \frac{1}{2019\cdot (1009)!}\cdot (2020)!$$

(c) Direct computation is possible but it is better to find the Maclaurin series for $\tan x$ first.

$$h(x) = \tan x,\ h(0) = 0$$

$$h'(x) = \sec^2 x,\ h'(0) = 1$$

$$h''(x) = 2\sec^2 x\tan x,\ h''(0) = 0$$

$$h'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x,\ h'''(0) = 2$$

$$h^{(4)}(x) = 8\sec^2 x\tan^4 x + 8\sec^4 x\tan x + 8\sec^4 x\tan x,\ h^{(4)}(0) = 0$$

$$h^{(5)}(x) = 16\sec^2 x\tan^6 x + 32\sec^4 x\tan^3 x + 64\sec^4 x\tan^2 x + 16\sec^6 x,\ h^{(5)}(0) = 16$$

$$\tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \cdots$$

$$f(x) = 1 + \frac{2}{3!}x^2 + \frac{16}{5!}x^4 + \cdots$$

Long division method: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots}$$

$$= x + \frac{0 + (-\frac{1}{6} + \frac{1}{2})x^3 + (\frac{1}{120} - \frac{1}{24})x^5 + (-\frac{1}{5040} + \frac{1}{720})x^7 + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots} = x + \frac{\frac{2}{3!}x^3 + \frac{-4}{5!}x^5 + \frac{6}{7!}x^7 + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots}$$

$$= x + \frac{2}{3!}x^3 + \frac{0 + (\frac{-4}{5!} + \frac{2}{3!}\frac{1}{2})x^5 + (\frac{6}{7!} - \frac{2}{3!}\frac{1}{24})x^7 + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots} = x + \frac{2}{3!}x^3 + \frac{\frac{16}{5!}x^5 + \frac{-64}{7!}x^7 + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots}$$

$$\tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \cdots$$

$$f(x) = 1 + \frac{2}{3!}x^2 + \frac{16}{5!}x^4 + \cdots$$

(d)

$$\lim_{x\to 0}\frac{x^2 f(x) - g(x)}{x^6} = \frac{16}{5!} - \frac{1}{10} = \frac{1}{30}$$

Grading scheme:

(a) (4%) 1 point to each process: (1) Starting function's Maclaurin series. (2) Plug in t^2. (3) Integrate. (4) Multiply.

(b) (3%) 1 point to formula, 1 point to finding correct n (depends on their answer from (a)), 1 point for final answer. [essentially, 1 point for each part of the answer]

(c) (5%) 1 point for the zero terms, 1 point for having a correct method, 1 point for each nonzero term. (For example: they lose 2 points if they get a nonzero value for a zero term in the answer, and loses 1 more if the reason they got a nonzero value is because of a wrong method.)

(d) (3%) Depends on the previous answers. 1 point for knowing the method (even if they have nothing in (a) and (c)), 1 point each for the coefficients (no points if these are not from (a) and (c)).

Alternatively, they can plug in the functions into (d) and use l'Hospital's Rule (3 points for correct answer, all or nothing).

4. (10%)

(a) (4%) Given $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = \ln 2$, by the Alternating Series Estimation Theorem, determine how many terms of the series do we need to estimate $\ln 2$ with error less than 0.01.

(b) (6%) We also know that $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)2n} = \ln 2$. By the Remainder Estimate for the Integral Test, determine how many terms of the series do we need to estimate $\ln 2$ with error less than 0.01.

Solution:

(a) By the Alternating Series Estimation theorem, $|R_N| \le |\frac{1}{N+1}| = \frac{1}{N+1}$.**(3pts)** We need to find n such that $\frac{1}{N+1} < 0.01$. Therefore, $n > 99$. 100 terms is enough.**(1pt)**

(b) Compare this with $f(x) = \frac{1}{2x(2x-1)}$. Since $f(x)$ is a decreasing function, from the Remainder Estimate for the Integral Test, we have $R_N < \int_N^{\infty} f(x)dx$**(3pts)**

$$R_N < \int_N^{\infty} \frac{1}{2x(2x-1)}dx = \frac{1}{2}\ln(\frac{2x-1}{2x})\Big|_N^{\infty} = \frac{1}{2}\ln(\frac{2N}{2N-1})$$

Correct integration worth **(2pts)**. If we want the error to be less than 0.01. Solve $\frac{1}{2}\ln(\frac{2N}{2N-1}) < 0.01$. We have

$$1 + \frac{1}{2N-1} = \frac{2N}{2N-1} < e^{0.02}.$$

We have $N > \frac{1}{2}(\frac{1}{e^{0.02}-1} + 1)$. Therefore, $[\frac{1}{2}(\frac{1}{e^{0.02}-1} + 1)] + 1$ terms is enough. **1pt** for determine N from the error 0.01.

(Note, from Taylor expansion, we have $e^{0.02} - 1 > 0.02$, $\frac{1}{2}(\frac{1}{e^{0.02}-1} + 1) < \frac{1}{2}(50+1) = \frac{51}{2}$. Choose any number $N > 26$ gives the desired result. This is not sharp.)

Grading policy:

- For part (b), checking $f(x)$ is positive and decreasing is **not** needed for obtaining full credits.
- For part (b), if the integration is wrong, the student may still earn **1pt** for solving N in terms of $R_N < 0.01$. The grader may decide depend on the work.
- For both parts, sharp bound in not required. Student only need to give the reasoning why the bound works from the indicated theorems.
- If there are some minor error such as using $\frac{1}{N}$ instead of $\frac{1}{N+1}$ in (a) or using $\int_{N+1}^{\infty} f(x)dx$ in (b), the student will lose at most 1 point from all these minor errors.

5. (8%) Let $\mathbf{F}(x,y,z) = yz\cos(xy)\,\mathbf{i} + xz\cos(xy)\,\mathbf{j} + \left(\sin(xy) + \dfrac{1}{1+z^2}\right)\mathbf{k}$.

(a) (4%) Compute $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$.

(b) (4%) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any curve from $(0,0,0)$ to $(3,3,1)$.

Solution:

(a) Set $P(x,y,z) = yz\cos(xy), Q(x,y,z) = xz\cos(xy)$ and $R(x,y,z) = \sin(xy) + \dfrac{1}{1+z^2}$.

(1) $\operatorname{curl}\mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ (1 point)

(2) Some calculation implies $\operatorname{curl}\mathbf{F} = (0,0,0)$ (1 point)

(3) $\operatorname{div}\mathbf{F} = P_x + Q_y + R_z$ (1 point)

(4) Some calculation implies $\operatorname{div}\mathbf{F} = -z\sin(xy)\cdot(x^2+y^2) - \dfrac{2z}{(1+z^2)^2}$ (1 point)

(b) (1) Since $\operatorname{curl}\mathbf{F} = (0,0,0)$ in $\mathbf{R}^3$, there exists a function f such that $\nabla f = F$. (1 point)

(2) Using partial integrations, we can find $f = z\sin(xy) + \arctan z + \textit{constant}$. (1 point)

(3) Applying fundamental theorem for line integral, we get $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,3,1) - f(0,0,0)$ (1 point).

(4) So $\int_C \mathbf{F}\cdot d\mathbf{r} = \sin 9 + \dfrac{\pi}{4}$. (1 point)

Remark:

(1) If you have the right answer for (b) just with respect to a special curve C, then you can get 2 points.

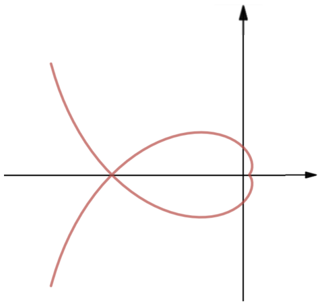
(2) If you have no (b)(1) and (b)(2) is right,then you also can get 2 points.

6. (16%) Let $\mathbf{F}(x,y) = \left(2x - \frac{2y}{x^2+y^2}\right)\mathbf{i} + \left(2y + \frac{2x}{x^2+y^2}\right)\mathbf{j}$.

(a) (5%) Determine whether $\mathbf{F}$ is conservative on the left half plane $x < 0$. If $\mathbf{F}$ is conservative, find a scalar potential function of $\mathbf{F}$.

(b) (5%) Evaluate directly $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}$ where C_1 is a circle $x^2+y^2=a^2$, for some $a>0$, oriented counterclockwise.

(c) (6%) Evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where C_2 is the polar curve $r = e^{|\theta|}$, $-\frac{7\pi}{6} \le \theta \le \frac{7\pi}{6}$, in the direction of increasing θ.



Solution:

(a) Let $P(x,y) = 2x - \frac{2y}{x^2+y^2}$ and $Q(x,y) = 2y + \frac{2x}{x^2+y^2}$. We compute that

$$\frac{\partial P}{\partial y} = -\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = -\frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2)}.\text{(1 point)}.$$

Since the left half plane $x<0$ is simply connected and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the left half plane $x<0$, $\mathbf{F}$ is conservative on the left half plane $x<0$.(1 point) (If you just check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the left half plane $x<0$, you get 2 points.)

Let $f(x,y)$ be a potential function of $\mathbf{F}$. We have that

$$f_x = 2x - \frac{2y}{x^2+y^2}$$

$$f_y = 2y + \frac{2x}{x^2+y^2}.$$

It implies that $f(x,y) = \int 2y + \frac{2x}{x^2+y^2}dy = x = y^2 + 2\tan^{-1}(\frac{y}{x}) + g(x)$ (1 point). Then we differentiate $f(x,y)$ w.r.t x to obtain that $f_x(x,y) = \frac{-2y}{x^2+y^2} + g'(x)$. So $g'(x) = 2x$(1 point). Then $g(x) = x^2 + C$ for some constant C. We take $C=0$. Therefore we find a potential function $f(x,y) = x^2 + 2\tan^{-1}(\frac{y}{x}) + y^2$ of $\mathbf{F}$. (If you directly find a potential function f of $\mathbf{F}$ and obtain correct answer, you get 5 points.)

(b) Let C_1 be represented by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$ where $0\le t\le 2\pi$(1 point). Thus

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(a\cos t, a\sin t)\cdot \mathbf{r}'(t)dt$$

$$= \int_0^{2\pi} \langle a\cos t - \frac{2a\sin t}{a^2}, a\sin t + \frac{2a\cos t}{a^2}\rangle \cdot \langle -a\sin t, a\cos t\rangle dt \text{ (2 points)}$$

$$= \int_0^{2\pi} (-a^2\sin t\cos t + 2\sin^2 t + a^2 \sin t\cos t + 2\cos^2 t)dt \text{ (1 point)}$$

$$= \int_0^{2\pi} 2dt = 4\pi \text{ (1 point)}.$$

(c) We take a circle $C_3 : x^2+y^2=a^2$ for some $a>0$ with counterclockwise direction such that C_3 is in the region which is enclosed by C_2. Let C_4 be the curve $r = e^{|\theta|}$ where $-\pi \le \theta \le \pi$ in the direction of increasing

θ. Let D be the region enclosed by $-C_3$ and C_4. Since F is conservative on the region D, using Green's Theorem, we have that

$$0 = \int_{-C_3 \cup C_4} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$
$$\Rightarrow \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi \text{ (3 points).}$$

Let C_5 be the tail of C_2 which is $r = e^{|\theta|}$ where $-\frac{7\pi}{6} \leq \theta \leq -\pi$ and $\pi \leq \theta \leq \frac{7\pi}{6}$. The tail C_5 is on the left half $x < 0$ and $\mathbf{F}$ is conservative on the left half plane $x < 0$. Thus $\int_{C_5} \mathbf{F} \cdot d\mathbf{r}$ is independent of path, which equals to

$$\begin{aligned} & f(e^{7\pi/6}\cos(7\pi/6), e^{7\pi/6}\cos(7\pi/6)) - f(e^{7\pi/6}\cos(-7\pi/6), e^{7\pi/6}\sin(-7\pi/6)) \\ = \; & 2\tan^{-1}(\tan\frac{7\pi}{6}) - 2\tan^{-1}(\tan\frac{-7\pi}{6}) \\ = \; & \frac{2\pi}{3}. \text{ (2 points)} \end{aligned}$$

Therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 4\pi + \frac{2\pi}{3} = \frac{14\pi}{3}$ (1 point).

Another Method: Let γ_1 be the curve $0 \leq \theta \leq \frac{7\pi}{6}$ which is a part of C_2 and γ_2 be the curve $-\frac{7\pi}{6} \leq \theta \leq 0$ which is a part of C_2 (1 point). Then

$$\begin{aligned} & \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ = \; & \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r} \\ = \; & \int_0^{7\pi/6} \langle 2e^\theta\cos\theta - \frac{2\sin\theta}{e^\theta}, 2e^\theta\sin\theta + \frac{2\cos\theta}{e^\theta}\rangle \cdot \langle e^\theta(\cos\theta - \sin\theta), e^\theta(\sin\theta + \cos\theta)\rangle d\theta \\ & + \int_{-7\pi/6}^{0} \langle 2e^{-\theta}\cos\theta - \frac{2\sin\theta}{e^{-\theta}}, 2e^{-\theta}\sin\theta + \frac{2\cos\theta}{e^{-\theta}}\rangle \\ & \cdot \langle -e^{-\theta}(\cos\theta + \sin\theta), e^{-\theta}(\cos\theta - \sin\theta)\rangle d\theta \text{ (1 point)} \\ = \; & \int_0^{7\pi/6} (2e^{2\theta} + 2) d\theta + \int_{-7\pi/6}^{0} (-2e^{-2\theta} + 2) d\theta \text{ (2 points)} \\ = \; & 2 \cdot 2 \cdot \frac{7\pi}{6} = \frac{14\pi}{3} \text{ (2 points).} \end{aligned}$$

7. (24%) In this question, you are given

a solid $V = \{(x, y, z) : x^2 + y^2 + z^2 \le 2 \text{ and } z \ge \sqrt{x^2 + y^2}\}$,

a surface $S_1 = \{(x, y, z) : x^2 + y^2 + z^2 \le 2 \text{ and } z = \sqrt{x^2 + y^2}\}$,

a surface $S_2 = \{(x, y, z) : x^2 + y^2 + z^2 = 2 \text{ and } z \ge \sqrt{x^2 + y^2}\}$, and

a vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (z + \sqrt{2 - x^2 - y^2})\,\mathbf{k}$.

Suppose both surfaces S_1 and S_2 are endowed with upward orientation.

(a) (8%) Evaluate, directly, the surface integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

(b) (8%) Evaluate, directly, the surface integral $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

(c) (5%) Evaluate, directly, the triple integral $\iiint_V \operatorname{div}\mathbf{F}\, dV$.

(d) (3%) Explain how your findings above are consistent with the Divergence Theorem.

Solution:

※ 如果沒有仔細寫出 parametrization，但是曲面積分列式正確，自動給 parametrization 的 2+1 分。

※ 如果曲面積分轉為參數積分轉換錯誤，但是不是嚴重錯誤，例如少乘 $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$，只扣轉化積分的 2pts，後面的重積分獨立給分。

※ 盡量每段落獨立給分。

(a) sol 1:
Parametrize S_1 by $\mathbf{r}(x, y) = (x, y, \sqrt{x^2 + y^2})$ where $(x, y) \in D = \{(x, y) | x^2 + y^2 \le 1\}$. (2pts for parametrization)

$\mathbf{r}_x \times \mathbf{r}_y = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right)$ (1pt for correct $\mathbf{r}_x \times \mathbf{r}_y$ or $\mathbf{n}$), $\left(\mathbf{n} = \frac{1}{\sqrt{2}} \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right) \right)$.

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS &= \iint_D \mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) dxdy \\
&= \iint_D \sqrt{2 - x^2 - y^2} dxdy \quad \text{(2pts for transforming the surface integral into an integral in parameters)} \\
D &= \{(r, \theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\} \\
&= \int_0^{2\pi} \int_0^1 \sqrt{2 - r^2} \cdot r\, dr\, d\theta = \left(-\frac{1}{3}(2 - r^2)^{\frac{3}{2}} \Big|_0^1 \right) \times 2\pi = \frac{2}{3}\pi(2\sqrt{2} - 1)
\end{aligned}$$

(3pts for correct integration)

sol 2:
Parametrize S_1 by $\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, r)$ where $0 \le r \le 1$, $0 \le \theta \le 2\pi$. (2pts for correct parametrization)

$\mathbf{r}_r \times \mathbf{r}_\theta = (-r\cos\theta, -r\sin\theta, r)$ pointing upward. (1pt for correct $\mathbf{r}_r \times \mathbf{r}_\theta$ or $\mathbf{n}$)

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 \mathbf{F}(\mathbf{r}(r, \theta)) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, drd\theta = \int_0^{2\pi} \int_0^1 (r\cos\theta, r\sin\theta, r + \sqrt{2 - r^2}) \cdot (-r\cos\theta, -r\sin\theta, r)\, drd\theta \\
&= \int_0^{2\pi} \int_0^1 \sqrt{2 - r^2} \cdot r\, drd\theta \\
&= \int_0^{2\pi} \int_0^1 \sqrt{2 - r^2} \cdot r\, drd\theta \text{ (3pts for transforming the surface integral into an integral in parameters)} \\
&= \frac{2}{3}\pi(2\sqrt{2} - 1) \text{ (2pts for correct integration)}
\end{aligned}$$

(b) Parametrize S_2 by $\mathbf{r}(\phi,\theta) = (\sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, \sqrt{2}\cos\phi)$ where $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$. (2pts for parametrization)

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ pointing upward. (1pt for correct $\mathbf{r}_\phi \times \mathbf{r}_\theta$ or $\mathbf{n}$)

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 2\sin\phi$, $\mathbf{n}(x,y,z) = \frac{1}{2}(x,y,z)$

3pts for transforming the surface integral

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS &= \iint_{S_2} (x, y, z+\sqrt{2-x^2-y^2})\cdot\frac{1}{\sqrt{2}}(x,y,z)\,dS \\ &= \iint_{S_2} \sqrt{2}+\frac{z^2}{\sqrt{2}}\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} (\sqrt{2}+\sqrt{2}\cos^2\phi)\cdot|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\phi d\theta \\ &= 2\sqrt{2}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}(1+\cos^2\phi)\sin\phi\,d\phi d\theta = \frac{\pi}{3}(16\sqrt{2}-14)\end{aligned}$$

(2pts for correct integration)

(c) $\text{div}\mathbf{F} = 1+1+1 = 3$ (1pt for $\text{div}\mathbf{F}$)

sol 1: Write V as type 1

$\{(x,y,z)|(x,y)\in D, \sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2}\}$ where D is the unit disc $\{(x,y)|x^2+y^2\le 1\}$ (1pt)

$$\begin{aligned}\iiint_V \text{div}\mathbf{F}\,dV &= \iint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} 3\cdot dz\,dA \\ &= 3\iint_D (\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2})\,dxdy \quad \text{(1pt)} \\ &= 3\int_0^{2\pi}\int_0^1(\sqrt{2-r^2}-r)r\,dr\,d\theta = 4\pi(\sqrt{2}-1) \quad \text{(2pts)}\end{aligned}$$

sol: 2

The spherical coordinates of V are described as $\{(\rho,\theta,\phi)|0\le\rho\le\sqrt{2}, 0\le\theta\le 2\pi, 0\le\varphi\le\frac{\pi}{4}\}$ (2pts)

$$\begin{aligned}\iiint_V \text{div}\mathbf{F}\,dV &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{2}} 3\rho^2\sin\phi\,d\rho\,d\theta\,d\varphi \quad \text{(1pt for Jacobian)} \\ &= 2\sqrt{2}\times 2\pi\times\left(1-\frac{1}{\sqrt{2}}\right) \\ &= 4\pi(\sqrt{2}-1) \quad \text{(1pt for final answer)}\end{aligned}$$

(d) The divergence theorem says that

$\iiint_V \text{div}\mathbf{F}\,dV = \iint_{\partial V}\mathbf{F}\cdot dS$ (1pt) $= \iint_{S_2}\mathbf{F}\cdot\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot\mathbf{S}$ (1pt for correct signs)

Indeed, $4\pi(\sqrt{2}-1) = \frac{\pi}{3}(16\sqrt{2}-14) - \frac{2\pi}{3}(2\sqrt{2}-1)$ (1pt)

8. (8%) To celebrate the 90th anniversary of NTU, a balloon of surface S for which the opening is the unit circle $x^2 + y^2 = 1$ on the xy-plane is erected at the main entrance (see Figure 1).

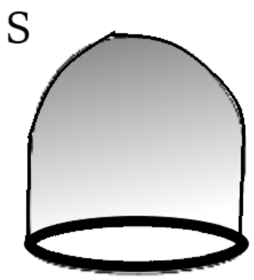


Figure 1

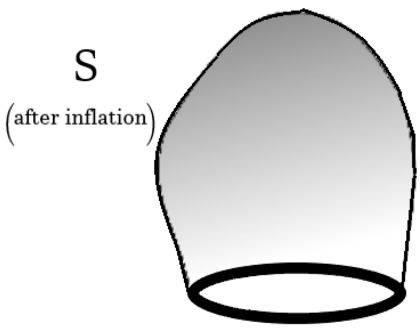


Figure 2

The velocity field of the air in the balloon is $\mathbf{G} = \text{curl}(\mathbf{F})$ where

$$\mathbf{F} = (\sin(xz) - y^3)\,\mathbf{i} + (x^3 - yz\cos(xz))\,\mathbf{j} + (z + e^y)\,\mathbf{k}.$$

(a) (5%) Compute the flux of $\mathbf{G}$ across S, oriented away from the origin.

(b) (3%) Now the balloon is further inflated while its opening remains the same (see Figure 2). Student A says '*If the surface area of S increases, then the flux of* $\mathbf{G}$ *across S also increases.*'. Do you agree with Student A ? Explain it mathematically.

Solution:

(a)

Marking scheme for Q8(a)

1M - correctly applying the Stokes'/Divergence Theorem

1M - on the correct parametrisation of the curve/ the surface

2M - setting up the correct line/surface integral with respect to the candidate's choice of parametrisation

1M - correct numerical answer

Partial credits for Q8(a)

- No marks will be awarded to candidates who only wrote 'by Stokes'/Divergence Theorem' without any elaboration.
- At most 1M is taken off if a candidate made an error in sign/orientation
- At most 3M can be awarded to candidates who parametrised the required curve/ surface incorrectly
- At most 1M can be awarded to candidates who assumed S is of a particular shape (say hemisphere/paraboloid).

Sample Solutions :

(Method 1) By Stokes' Theorem, we have $\iint_S \mathbf{G} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle $x^2 + y^2 = 1$ on xy-plane anticlockwisely oriented.(1M)

Parametrise C by $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ (1M) with $0 \le t \le 2\pi$.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\sin^3 t, \cos^3 t, \star \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \ (2M) \\ &= \int_0^{2\pi} \sin^4 t + \cos^4 t \, dt \\ &= \frac{3\pi}{2} \ (1M)\end{aligned}$$

(Method 2) By Stokes' Theorem, $\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_D \mathbf{G}\cdot d\mathbf{S}$ where D is the unit disk $x^2+y^2 \le 1$ on xy-plane, oriented upward (1M).

Parametrise D by $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ (1M) with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Note that the $\mathbf{k}$-component of $\mathbf{G}$ equals to $3x^2 + 3y^2 + yz^2\sin(xz)$.

$$\begin{aligned}\iint_D \mathbf{G}\cdot d\mathbf{S} &= \iint_D \langle \star, \star, 3r^2\rangle \cdot \langle 0,0,1\rangle r\,dr\,d\theta (2M)\\ &= \int_0^{2\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta \\ &= \frac{3\pi}{2}. (1M)\end{aligned}$$

(Method 3) Let D be the unit disk $x^2+y^2 \le 1$ on xy-plane, oriented upward and U be the solid enclosed by S and D. By Divergence Theorem, $\iint_S \mathbf{G}\cdot d\mathbf{S} - \iint_D \mathbf{G}\cdot d\mathbf{S} = \iiint_U \operatorname{div}\mathbf{G}\,dV = 0$. (1M) The rest is identical to Method 2 (4M).

(b)

Marking scheme for Q8(b)

1M - Mentioning Stokes' or Divergence Theorem

2M - Correct explanation (the keyword is 'same (oriented) boundary')

Partial credits for Q8(b)

- No marks will be awarded to candidates who just agree/disagree with Student A without any reasonable, mathematical argument.
- At most 1M will be given to candidates who just say 'I disagree with Student A because of Stokes'/ Divergence Theorem' without an accurate elaboration.

Sample Solutions :

(Method 1) By Stokes' Theorem (1M), the flux of $\mathbf{G} = \operatorname{curl}(\mathbf{F})$ is the same across any surfaces with the same (oriented) boundary (2M). Therefore, we don't agree with Student A.

(Method 2) Since $\operatorname{div}\mathbf{G} = 0$, the Divergence Theorem (1M) implies that the flux of S, despite inflated, equals to that across the unit disk on xy-plane (oriented upward) (2M). Therefore, we don't agree with Student A.