1081模組03-07班期中考解答和評分標準

1. (12 pts) Find the following limits.

(a) (5 pts) $\lim_{x\to\infty} \frac{\sqrt{2x^8+4x+1}+5x^3}{x^4+1}$.

(b) (7 pts) $\lim_{x\to 1} \tan^{-1}\left(\frac{4\sqrt{x}-4}{x^2-1}\right)$.

Solution:

(a)

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{2x^8+4x+1}+5x^3}{x^4+1} &= \lim_{x\to\infty} \frac{\sqrt{2+\frac{4}{x^7}+\frac{1}{x^8}}+\frac{5}{x}}{1+\frac{1}{x^4}} \quad \boxed{4\text{ pt}}\\ &= \frac{\lim_{x\to\infty}\sqrt{2+\frac{4}{x^7}+\frac{1}{x^8}}+\lim_{x\to\infty}\frac{5}{x}}{1+\lim_{x\to\infty}\frac{1}{x^4}}\\ &= \sqrt{2}. \quad \boxed{1\text{ pt}}\end{aligned}$$

(b) We first calculate

$$\lim_{x\to 1}\frac{4\sqrt{x}-4}{x^2-1} = \begin{cases} \lim_{x\to 1}\frac{4(\sqrt{x}-1)}{(x+1)(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x\to 1}\frac{4}{(x+1)(\sqrt{x}+1)} \\ \lim_{x\to 1}\left(\frac{4(\sqrt{x}-1)}{x^2-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1}\frac{4(x-1)}{(x^2-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{4}{(x+1)(\sqrt{x}+1)} \\ \lim_{x\to 1}\frac{\frac{2}{\sqrt{x}}}{2x} = \lim_{x\to 1}\frac{1}{x\sqrt{x}} \quad \text{(by using L'Hospital rule)} \end{cases}$$

$\boxed{\text{each is 4 pt}}$

$$= 1. \quad \boxed{1\text{ pt}}$$

Note that $y = \tan^{-1} x$ is a continuous function. Hence,

$$\begin{aligned}\lim_{x\to 1}\tan^{-1}\left(\frac{4\sqrt{x}-4}{x^2-1}\right) &= \tan^{-1}\left(\lim_{x\to 1}\frac{4\sqrt{x}-4}{x^2-1}\right) \quad \boxed{1\text{ pt}}\\ &= \tan^{-1} 1\\ &= \frac{\pi}{4}. \quad \boxed{1\text{ pt}}\end{aligned}$$

2. (12 pts) Find the following limits.

(a) (5 pts) $\lim_{x\to 0} \dfrac{\tan^{-1}(a\,x)}{\tan^{-1}(b\,x)}$, where a and $b \neq 0$ are constants.

(b) (7 pts) $\lim_{x\to 0}(1+\sin 2x)^{\frac{1}{3x}}$.

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan^{-1}(ax)}{\tan^{-1}(bx)} &= \lim_{x\to 0}\frac{(\tan^{-1}(ax))'}{(\tan^{-1}(bx))'}(\text{ Type } \frac{0}{0},\ \text{using L'Hospital's Rule})(\text{ 1 point})\\
&= \lim_{x\to 0}\frac{\frac{1}{1+a^2x^2}\cdot a}{\frac{1}{1+b^2x^2}\cdot b}(\text{2 points})\\
&= \lim_{x\to 0}\frac{a(1+b^2x^2)}{b(1+a^2x^2)}(\text{ 1 point})\\
&= \frac{a}{b}(\text{ 1 point}).
\end{aligned}$$

(b) We have that $\ln(1+\sin 2x)^{\frac{1}{3x}} = \dfrac{1}{3x}\ln(1+\sin 2x)$. (2 points)

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(1+\sin 2x)}{3x} &= \lim_{x\to 0}\frac{\frac{2\cos 2x}{1+\sin 2x}}{3}(\text{Type}\frac{0}{0},\ \text{Using L'Hospital's Rule})(\text{ 2 points})\\
&= \frac{2}{3}(\text{ 1 point}).
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{x\to 0}(1+\sin 2x)^{\frac{1}{3x}} &= e^{\lim_{x\to 0}\frac{1}{3x}\ln(1+\sin 2x)}\\
&= e^{2/3}(\text{ 2 points}).
\end{aligned}$$

Another Method

$$\begin{aligned}
\lim_{x\to 0}(1+\sin 2x)^{\frac{1}{3x}} &= \lim_{x\to 0}(1+\sin 2x)^{\frac{1}{\sin 2x}\cdot\frac{\sin 2x}{3x}}\\
&= \lim_{x\to 0}\left[(1+\sin 2x)^{\frac{1}{\sin 2x}}\right]^{\frac{\sin 2x}{3x}}\\
&= e^{2/3}
\end{aligned}$$

because $\lim_{x\to 0}(1+\sin 2x)^{\frac{1}{\sin 2x}} = e$ and $\lim_{x\to 0}\dfrac{\sin 2x}{3x} = 2/3$.

3. (11 pts) Find the derivative of $f(x)$.

(a) (5 pts) $f(x) = x^{\frac{4}{3}} + x \cdot 2^{(x^2+1)}$.

(b) (6 pts) $f(x) = x \cdot \sec^{-1} x - \frac{1}{2} \ln\left(x^2+1\right)$.

Solution:

(a)

$$f'(x) = \underbrace{\frac{4}{3}x^{\frac{1}{3}}}_{\text{1 pt}} + \underbrace{\left[2^{(x^2+1)} + x \cdot (2^{(x^2+1)})'\right]}_{\text{1 pt(For product rule)}} = \frac{4}{3}x^{\frac{1}{3}} + 2^{(x^2+1)} + x \cdot \underbrace{\ln 2}_{\text{1 pt}} \cdot \underbrace{2^{(x^2+1)}}_{\substack{\text{1 pt} \\ \text{(derivative of } 2^x)}} \cdot \underbrace{2x}_{\substack{\text{1 pt} \\ \text{(chain rule)}}}$$

$$= \frac{4}{3}x^{1/3} + 2^{(x^2+1)} + 2\ln 2 \cdot x^2 \cdot 2^{x^2+1}$$

(b)

$$f'(x) = \underbrace{1 \cdot \sec^{-1} x + x \cdot \frac{1}{x\sqrt{x^2-1}}}_{\substack{\text{product rule 1 pt} \\ (\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}} \text{ 2 pts}}} - \underbrace{\frac{1}{2}\frac{1}{x^2+1}}_{\substack{\text{1 pt} \\ \text{(derivative of } \ln x)}} \cdot \underbrace{2x}_{\text{1 pt(chain rule)}}$$

$$= \sec^{-1} x + \frac{1}{\sqrt{x^2-1}} - \frac{x}{x^2+1}$$

4. (12 pts)

(a) (6 pts) $f(x) = (x^2+1)^{\cos x}$. Find $f'(x)$.

(b) (6 pts) Find the equation of the tangent line to the curve satisfying $x^{\frac{2}{3}} + y^{\frac{2}{3}} + y = 6$ at $(8, 1)$.

Solution:

(a) $\ln f(x) = \cos x \ln(x^2+1) \xrightarrow{\frac{d}{dx}} \dfrac{f'(x)}{f(x)} = -\sin x \ln(x^2+1) + \cos x \cdot \dfrac{2x}{x^2+1}$

$\Rightarrow f'(x) = f(x)[-\sin x \cdot \ln(x^2+1) + \cos x \cdot \dfrac{2x}{x^2+1}] = (x^2+1)^{\cos x}[-\sin x \cdot \ln(x^2+1) + \cos x \cdot \dfrac{2x}{x^2+1}]$

微分錯誤，一個錯誤扣1分。

(b) $\dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{2}{3}y^{-\frac{1}{3}}y' + y' = 0$

$(x, y) = (8, 1)$ 帶入得 $\dfrac{2}{3} \cdot \dfrac{1}{2} + (\dfrac{2}{3} + 1)y' = 0,\ y' = -\dfrac{1}{5}$

$\dfrac{y-1}{x-8} = -\dfrac{1}{5}$

- (3分)隱函數微分。
- (1分)以$y(x)$和x寫出$y'(x)$。
- (1分)計算$y'(1)$.
- (1分)寫出切線。

5. (16 pts) Consider the function

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x} + 2x & \text{if } x > 0, \\ ax + b & \text{if } x \le 0. \end{cases}$$

Suppose that f is differentiable everywhere.

(a) (8 pts) Find the values of a and b.

(b) (8 pts) Write down the linear approximation of $f(x)$, at $x = 0$. Use the linear approximation to estimate $f(0.01)$.

Solution:

(a) Step1: (1 point) By Squeeze Theorem we have that $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x + x^3 \sin \frac{1}{x} = 0$.

Step2: (3 points) Since f is differentiable everywhere, f is continuous everywhere (1 point). Thus we have that $\lim_{x \to 0^-} f(x) = b = \lim_{x \to 0^+} f(x) = 0$. (2 points)

Step3: (2 point) Since f is differentiable everywhere, f is differentiable at $x = 0$. So the following limit exists.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}.$$

Step 4: So we have that

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(h)}{h} &= \lim_{h \to 0^+} \frac{2h + h^3 \sin \frac{1}{h}}{h} \\ &= 2 (1 \text{ point}) \\ &\quad \text{by Squeeze Theorem, } \lim_{h \to 0^+} h^2 \sin \frac{1}{h} = 0. (1 \text{ points}) \\ \lim_{h \to 0^-} \frac{ah}{h} &= \lim_{h \to 0^-} a. \end{aligned}$$

Thus $a = 2$. Therefore if $a = 2$ and $b = 0$, then $f(x)$ is differentiable everywhere.

(b) Since $f(x)$ is differentiable at $x = 0$, the linear approximation of $f(x)$ at $x = 0$ is

$$\begin{aligned} L(x) &= f(0) + f'(0)(x - 0) (\text{ 4 points}) \\ &= 2x. (2 \text{ points}) \end{aligned}$$

Thus $f(0.01) \approx 2 * 0.01 = 0.02$ (2 points).

6. (18 pts) $f(x) = \dfrac{3x^2 - x + 2}{x - 2}$.

(a) (6 pts) Compute $f'(x)$ and find interval(s) of increase of $f(x)$ and interval(s) of decrease of $f(x)$. Find local extreme values of $f(x)$.

(b) (4 pts) Compute $f''(x)$ and find concavity and inflection points of $y = f(x)$.

(c) (4 pts) Find all vertical, horizontal and slant asymptotes of the curve $y = f(x)$.

(d) (4 pts) Sketch the graph of $f(x)$.

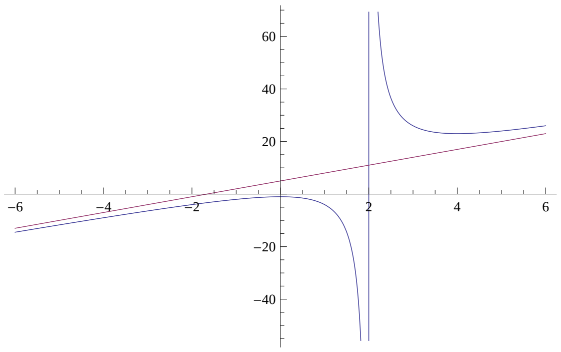
Solution:

(a) $f'(x) = \dfrac{3(x-4)x}{(x-2)^2}$ (2pts). $f(x)$ is increasing for $x > 4$ or $x < 0$ (2pts). $f(x)$ is decreasing for $0 < x < 4$ (2pts).

(b) $f''(x) = \dfrac{24}{(x-2)^3}$ (1pts). $f(x)$ is concave upward for $x > 2$ (1pts). $f(x)$ is concave downward for $x < 2$ (1pts). Inflection point: none (1pts).

(c) Vertical asymptotes: $x = 2$ (1pts). Horizontal asymptotes: none (1pts). Slant asymptotes: $f(x) = \dfrac{3x^2 - x + 2}{x - 2} = 5 + 3x + \dfrac{12}{x - 2} \implies y = 5 + 3x$ (2pts).

(d) The curve $y = f(x)$ (2pts). Vertical asymptotes $x = 2$ (1pts). Slant asymptotes $y = 5 + 3x$ (1pts).



7. (19 pts) A firm finds that the total cost $C(x)$(in dollars) of manufacturing x tennis rackets/day is given by $C(x) = 400 + 4x + 0.0001x^2$. Each racket can be sold at a price of p dollars related to x by the equation $p(x) = 12 - 0.0004x$.

(a) (5 pts) Find the daily level of production, x_1, that minimizes the average cost $\dfrac{C(x)}{x}$. (You need to check that the value you find is indeed the minimum value.)

(b) (4 pts) Show that the average cost, $\dfrac{C(x)}{x}$, equals the marginal cost $C'(x)$ when $x = x_1$.

(c) (5 pts) Find the daily level of production, x_2, that maximizes the profit $\Pi(x) = x \cdot p(x) - C(x)$. (You need to check that the value you find is indeed the maximum value.)

(d) (5 pts) Find the inverse function of $p(x) = 12 - 0.0004x$ which is denoted by $x = F(p)$. Find the point elasticity $\epsilon = \dfrac{F'(p) \cdot p}{F(p)}$. In the interval $p \in (0, 12)$, find values of p such that $-1 < \epsilon < 0$(inelastic) and values of p such that $\epsilon < -1$(elastic).

Solution:

(a)

Let $AC(x) = \dfrac{C(x)}{x} = \dfrac{400}{x} + 4 + 0.0001x$ (1 pt)

$(AC(x))' = -\dfrac{400}{x^2} + 0.0001$ (1 pt)

For $x \in (0, 2000)$, $(AC(x))' < 0$ and $(AC(x))' > 0$ for $x \in (2000, \infty)$ (2 pts First derivative test)

Hence $AC(x)$ obtains minimum value at $x_1 = 2000$ (1 pt)

(b)

sol 1

$\dfrac{C(2000)}{2000} = 0.2 + 4 + 0.2 = 4.4$ (1 pt)

$C'(x) = 4 + 0.0002x$ (1 pt)

$C'(2000) = 4.4$ (1 pt)

Hence at $x_1 = 2000$, $\dfrac{C(x)}{x}$ equals $C'(x)$ (1 pt)

sol 2

(1 pt) $\because AC(x) = \dfrac{C(x)}{x}$ is differentiable

$\therefore$ At the minimum value $x = x_1$, $AC'(x_1) = 0$ by Fermat's Theorem

(1 pt) However $AC'(x) = \dfrac{C'(x) \cdot x - C(x)}{x^2}$

(2 pts) $AC'(x_1) = 0 \Rightarrow C'(x_1) \cdot x_1 - C(x_1) = 0 \Rightarrow C'(x_1) = \dfrac{C(x_1)}{x_1}$

(c)

(2 pts) $\Pi(x) = x \cdot p(x) - c(x) = x \cdot (12 - 0.0004x) - 400 - 4x - 0.0001x^2$
$\Pi'(x) = 12 - 0.0008x - 4 - 0.0002x = 8 - 0.001x$

(2 pts) $\Pi'(x) > 0$ for $0 < x < 8000$
$\Pi'(x) < 0$ for $x > 8000$

(1 pt) Hence $\Pi(x)$ obtains maximum value at $x_2 = 8000$

(d) (2 pts)(The inverse function of $p(x)$)

$p = 12 - 0.0004x \Leftrightarrow x = \dfrac{1}{0.0004}(12 - p) = 2500(12 - p)$ i.e. $F(p) = 2500(12 - p)$

(2 pts)(computation of ϵ)

$\epsilon(p) = \dfrac{F'(p) \cdot p}{F(p)} = \dfrac{-2500 \cdot p}{2500(12 - p)} = \dfrac{-p}{12 - p}$

For $0 < p < 12$, $12 \cdot p > 0$ and

(2 pts) $\begin{array}{l} -1 < \epsilon < 0 \Leftrightarrow -1(12 - p) < -p < 0 \Leftrightarrow 0 < p < 6 \\ \epsilon < -1 \Leftrightarrow -p < -(12 - p) \Leftrightarrow 12 > p > 6 \end{array}$

Ans: $-1 < \epsilon < 0$ for $0 < p < 6$. $\epsilon < -1$ for $6 < p < 12$.