1. (6 pts) Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_4^{x^2} \frac{t}{1+t^4}\,\mathrm{d}t$.

Solution:
We have $f'(x) = e^{g(x)}g'(x)$(1 point). From Fundamental Theorem of Calculus (2 point),

$$g'(x) = \frac{d}{dx}\int_4^{x^2} \frac{t}{1+t^4}dt = \frac{x^2}{1+x^8}\cdot 2x = \frac{2x^3}{1+x^8}\text{(1 point)}.$$

We get $e^{g(2)} = e^0 = 1$ (1 point) and $g'(2) = \frac{2^4}{1+2^8} = \frac{16}{257}$(1 point). Therefore $f'(2) = \frac{16}{257}$.

2. Compute the following integrals.

(a) (6 pts) $\int_0^1 \sin^{-1}(x)\,dx$ (b) (10 pts) $\int \sqrt{1+x^2}\,dx$ (c) (12 pts) $\int \frac{x^3+4x^2+4x+2}{x^4+2x^3+2x^2}\,dx$.

Solution:

(a) $\int_0^1 \sin^{-1} x dx = x\sin^{-1} x\Big|_0^1 - \int_0^1 x(\sin^{-1} x)' dx$ ⋯⋯⋯⋯(2pts for integration by parts)

$= \frac{\pi}{2} - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ ⋯⋯⋯⋯(1pt for $\sin^{-1} 1 = \frac{\pi}{2}$ and 1pt for the other)

$= \frac{\pi}{2} + \frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}}$ (Let $u = 1-x^2 \Rightarrow du = -2xdx$)⋯⋯⋯⋯(1pt for correct substitution)

$= \frac{\pi}{2} - \sqrt{u}\Big|_0^1 = \frac{\pi}{2} - 1$ ⋯⋯⋯⋯(1pt)

(b) $\int \sqrt{1+x^2} dx = \int \sqrt{\sec^2\theta}\sec^2\theta d\theta$ (Let $x = \tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow dx = \sec^2\theta d\theta$)

⋯⋯⋯⋯(3pts for trigonometric substitution)

$= \int \sec^3\theta d\theta$ ($\because \sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

$\int \sec^3\theta d\theta = \int \sec\theta\sec^2\theta d\theta = \sec\theta\tan\theta - \int \tan\theta(\sec\theta)' d\theta$ ⋯⋯⋯⋯(2pts for integration by parts)

$= \sec\theta\tan\theta - \int \sec\theta\cdot\tan^2\theta d\theta = \sec\theta\tan\theta - \int \sec^2\theta(\sec\theta - 1) d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int \sec^3\theta d\theta.$ (3pts)

Thus, $\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

Hence $\int \sqrt{1+x^2}\,dx = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\sqrt{1+x^2}\cdot x + \frac{1}{2}\ln|\sqrt{1+x^2}+x| + C$

(2 pts) ($\because \tan\theta = x,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\ \therefore \sec\theta = \sqrt{1+x^2}$)

(c) First, we factorize the denominator:

$$x^4+2x^3+2x^2 = x^2(x^2+2x+2).\text{(1 point)}$$

We write the integrand as follows

$$\frac{x^3+4x^2+4x+2}{x^4+2x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+2}\text{(2 point)}.$$

We have

$$x^3+4x^2+4x+2 = Ax(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)x^2 \tag{1}$$

Put $x = 0$ into Equation (1), we get $B = 1$.
Compare the coefficient of x in Equation (1), we have $2A+2B = 4 \Rightarrow A = 1$.
Compare the coefficient of x^2 in Equation (1), we have that $2A+B+D = 4 \Rightarrow D = 1$.
Compare the coefficient of x^3 in Equation (1), we have that $A+C = 1 \Rightarrow C = 0$.

Thus $\frac{x^3+4x^2+4x+2}{x^4+2x^3+2x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^2+2x+2}$(3 points). Since

$$\begin{aligned}\int \frac{1}{x}dx &= \ln|x| + C\text{(1 point)}\\ \int \frac{1}{x^2}dx &= -x^{-1} + C\text{(1 point)}\\ \int \frac{1}{x^2+2x+2}dx &= \int \frac{1}{1+(x+1)^2}dx = \tan^{-1}(x+1) + C\text{(2 points)},\end{aligned}$$

we have

$$\int \frac{x^3+4x^2+4x+2}{x^4+2x^3+2x^2}dx = \ln|x| - \frac{1}{x} + \tan^{-1}(x+1) + C.$$

3. (10 pts) Determine whether the following improper integral is convergent or divergent. Evaluate it if it is convergent.

$$\int_0^\infty e^{-2x} \cos x \, \mathrm{d}x \ .$$

Solution:

Let $u = e^{-2x}$ and $v = \sin x$, then

$$\mathrm{d}u = -2e^{-2x}\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}v = \cos x\,\mathrm{d}x \ .$$

Perform integration by parts

$$\int e^{-2x}\cos x\,\mathrm{d}x = e^{-2x}\sin x + 2\int e^{-2x}\sin x\,\mathrm{d}x \ .$$

1 pt: 第一次提到/使用 by parts, 1 pt: 操作正確

Similarly, for the last integral, let $\tilde{u} = e^{-2x}$ and $\tilde{v} = -\cos x$, then

$$\mathrm{d}\tilde{u} = -2e^{-2x}\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}\tilde{v} = \sin x\,\mathrm{d}x \ .$$

Again, use integration by parts

$$\int e^{-2x}\sin x\,\mathrm{d}x = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,\mathrm{d}x \ .$$

1 pt: 第二次提到/使用 by parts, 3 pt: 函數選取正確, 1 pt: 計算正確

Putting these together gives

$$\int e^{-2x}\cos x\,\mathrm{d}x = e^{-2x}\sin x - 2e^{-2x}\cos x - 4\int e^{-2x}\cos x\,\mathrm{d}x \ ,$$
$$\Rightarrow \int e^{-2x}\cos x\,\mathrm{d}x = \frac{1}{5}e^{-2x}(\sin x - 2\cos x) + C \ .$$

Therefore,

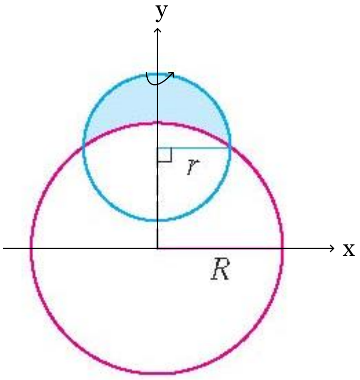
$$\int_0^t e^{-2x}\cos x\,\mathrm{d}x = \frac{1}{5}e^{-2t}(\sin t - 2\cos t) + \frac{2}{5} \ .$$

1 pt: 得到正確定積分

Note that $|\sin t - 2\cos t| \le 3$ for any t and $e^{-2t} \to 0$ as $t \to \infty$. As $t \to \infty$, the limit of the right hand side exists, and is equal to 2/5. That is to say, the improper integral converges, and is 2/5.

1 pt: 正確論證極限過程, 1 pt: 答案正確

4. (10 pts) Consider the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R, where $0 < r < R$. Rotate the region about the y-axis. Find the resulting volume.



Solution:

$$2\pi \int_0^r \left(\sqrt{r^2 - x^2} + \sqrt{R^2 - r^2} - \sqrt{R^2 - x^2}\right) x \,\mathrm{d}x = \frac{2}{3}\pi \left(r^3 - R^3 + (R^2 + \frac{r^2}{2})\sqrt{R^2 - r^2}\right)$$

5. (12 pts) Given an increasing supply function $S(q)$ and a decreasing demand function $D(q)$ where $S(q)$ and $D(q)$ are continuous, we define the total surplus at quantity q as $TS(q) = \int_0^q D(t) - S(t)\,dt$ for $q \geq 0$.

(a) (4 pts) Show that if $D(q^*) = S(q^*)$ for some $q^* > 0$ then $TS(q)$ obtains the absolute maximum value at $q = q^*$.

(b) (8 pts) Suppose that $D(q) = 6 - \left(1 + \frac{q}{2}\right)^{2/3}$ and $S(q) = \left(1 + \frac{q}{2}\right)^{1/3}$. Compute $TS(q)$ and find the absolute maximum value of $TS(q)$.

Solution:

(a) $TS'(q) = \frac{d}{dq}\int_0^q D(t) - S(t)dt = D(q) - S(q)$ ············(2pts)

Because $D(q)$ is decreasing and $S(q)$ is increasing, we know that $TS'(q) = D(q) - S(q)$ is increasing.

Hence $D(q^*) = S(q^*)$ implies that $TS'(q^*) = 0$. ············(1pt)

Moreover, $TS'(q) > 0$ for $0 < q < q^*$ and $TS'(q) < 0$ for $q > q^*$.

Hence $TS(q)$ obtains absolute maximum value at $q = q^*$. } ············(1pt)

(b)

$$TS(q) = \int_0^q 6 - \left(1+\frac{t}{2}\right)^{\frac{2}{3}} - \left(1+\frac{t}{2}\right)^{\frac{1}{3}} dt$$

$$\left(\text{Let } u = 1 + \frac{t}{2} \Rightarrow du = \frac{1}{2}dt, dt = 2du\right) = 2\int_1^{1+q} 6 - u^{\frac{2}{3}} - u^{\frac{1}{3}} du \text{········(2pts for correct substitution.)}$$

$$= 2\left[6u - \frac{5}{3}u^{\frac{3}{5}} - \frac{3}{4}u^{\frac{4}{3}}\right]_{u=1}^{u=1+\frac{q}{2}}$$

$$= 6q - \frac{6}{5}\left(1+\frac{q}{2}\right)^{\frac{5}{3}} - \frac{3}{2}\left(1+\frac{q}{2}\right)^{\frac{4}{3}} + \frac{27}{10} \text{············(2pts)}$$

From (a), we know that the absolute maximum value of $TS(q)$ occurs at $q = q^*$ if $D(q^*) = S(q^*)$ (1pt)

$D(q^*) = S(q^*) \Rightarrow 6 - \left(1 + \frac{q^*}{2}\right)^{\frac{2}{3}} = \left(1 + \frac{q^*}{2}\right)^{\frac{1}{3}}$.

Let $y = \left(1 + \frac{q^*}{2}\right)^{\frac{1}{3}}$. $6 - y^2 = y \Rightarrow y = 2$ or -3 ············(2pts)

$\because q^* > 0 \therefore \left(1 + \frac{q^*}{2}\right)^{\frac{1}{3}} = 2 \Rightarrow q^* = 14$

$TS(q^*) = 6 \times 14 - \frac{6}{5}\cdot 2^5 - \frac{3}{2}\cdot 2^4 + \frac{27}{10} = 24.3$ ············(1pt)

6. (a) (6 pts) Solve the initial-value problem: $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$.

(b) (10 pts) Solve the differential equation $(x^2+2)y'(x) + (4x)y = 2x$ with $y(0) = 2$.

Solution:

(a)

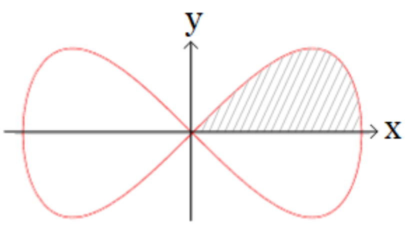
$$2udu = (2t + \sec^2 t)dt \quad \Rightarrow \int 2u\ du = \int (2t + \sec^2 t)\ dt\ (1pt)$$
$$\Rightarrow u^2 = t^2 + \tan t + C, \quad C \text{ is constant } (3pts)$$
$$\text{by initial condition } \Rightarrow C = 25\ (2pts)$$
$$\Rightarrow u^2 = t^2 + \tan t + 25 \Rightarrow u = -\sqrt{t^2 + \tan t + 25}$$

(b)

$$\Rightarrow \quad (x^2+2)\,y' + (4\,x)y = 2\,x$$
$$\Rightarrow \quad y'(x) + \frac{4\,x}{x^2+2}y = \frac{2\,x}{x^2+2}\ (2pts)$$
$$\Rightarrow \quad I(x) = e^{\int \frac{4\,x}{x^2+2}\ dx} = e^{(\ln(x^2+2)^2)} = (x^2+2)^2\ (4pts)$$
$$\Rightarrow \quad ((x^2+2)^2 y)' = 2\,x(x^2+2) \Rightarrow (x^2+2)^2 y = \frac{1}{2}x^4 + 2\,x^2 + C, C \text{ is constant } (2pts)$$
$$\text{by initial condition } \Rightarrow C = 8 \quad \Rightarrow y = \frac{x^4 + 4x^2 + 16}{2\,(x^2+2)^2}\ (2pts)$$

7. (18 pts) The *eight-like curve* has the following parametric equation:

$$x = 2\sqrt{2}\sin t\ ,\quad y = \sin t\cos t \quad \text{for} \quad 0 \le t \le 2\pi\ .$$



(a) (4 pts) Find the tangent line at $t = 0$.

(b) (8 pts) Find its arc length.

(c) (6 pts) Find the shaded area which is enclosed by the curve $0 \le t \le \frac{\pi}{2}$ and the x-axis.

Solution:

(a) The slope is given by

$$\frac{y'(t)}{x'(t)}\bigg|_{t=0} = \frac{\cos^2 t - \sin^2 t}{2\sqrt{2}\cos t}\bigg|_{t=0} = \frac{1}{2\sqrt{2}}\ .$$

[2 pts]: 知道斜率為 y'/x' (若微分計算錯誤，後續均不給分), [1 pt]: 正確求得該點斜率 (求得 $\pm\frac{1}{2\sqrt{2}}$ 不給分)

The position at $t = 0$ is $(x(0), y(0)) = (0, 0)$. It follows that the tangent line is $y = \frac{1}{2\sqrt{2}}x$. [1 pt]: 正確求得該點切線方程式 (求得 $y = \pm\frac{1}{2\sqrt{2}}x$ 不給分)

(b)

$$\begin{aligned}(x')^2 + (y')^2 &= (2\sqrt{2}\cos t)^2 + (\cos^2 t - \sin^2 t)^2\\ &= 8\cos^2 t + (2\cos^2 t - 1)^2\\ &= 8\cos^2 t + 4\cos^4 t - 4\cos^2 t + 1\\ &= (2\cos^2 t + 1)^2\ .\end{aligned}$$

[1 pt]: 知道要計算 $(x')^2 + (y')^2$, [1 pt]: 一個微分計算正確, [2 pts]: 整理成完全平方

The arc length is

$$\begin{aligned}\int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, \mathrm{d}t &= \int_0^{2\pi} (2\cos^2 t + 1)\, \mathrm{d}t\\ &= \int_0^{2\pi} (\cos(2t) + 2)\, \mathrm{d}t\\ &= (\frac{1}{2}\sin(2t) + 2t)\bigg|_{t=0}^{2\pi}\\ &= 4\pi\ .\end{aligned}$$

[1 pt]: 曲線弧長列式正確(含正確積分範圍), [2 pts]: 不定積分計算正確, [1 pt]: 最後帶值計算正確

(c) 解法一 As t goes from 0 to $\pi/2$, $x(t)$ is increasing. The area of the shaded region is

$$\int_0^{\frac{\pi}{2}} y(t)\, \mathrm{d}x(t) = \int_0^{\frac{\pi}{2}} (\sin t\cos t)(2\sqrt{2}\cos t\mathrm{d}t)\ .$$

[1 pt]: 知道要計算 $\int y\mathrm{d}x$, [1 pt]: 換為 $\mathrm{d}t$ 式子正確(含正確積分範圍)

Let $u = \cos t$, $\mathrm{d}u = -\sin t \mathrm{d}t$. The integral becomes

$$2\sqrt{2}\int_0^1 u^2 \mathrm{d}u = 2\sqrt{2}\,\frac{u^3}{3}\bigg|_{u=0}^{1} = \frac{2\sqrt{2}}{3}\ .$$

$\boxed{1\text{ pt}}$: 變數變換選取正確, $\boxed{1\text{ pt}}$: 變數變換範圍判斷正確(若先對不定積分變數變換, 則「不定積分計算正確」佔2分), $\boxed{1\text{ pt}}$: 不定積分計算正確, $\boxed{1\text{ pt}}$: 求值正確

<u>解法二</u> As t goes from 0 to $\pi/2$, x goes from 0 to $2\sqrt{2}$. The area of the shaded region is

$$\int_0^{2\sqrt{2}} y\,\mathrm{d}x = \int_0^{2\sqrt{2}} \frac{x}{2\sqrt{2}}\sqrt{1-\frac{x^2}{8}}\,\mathrm{d}x\ .$$

$\boxed{1\text{ pt}}$: 知道要計算 $\int y\mathrm{d}x$, $\boxed{2\text{ pts}}$: 定積分範圍以及 $y(x)$ 表達正確

Let $u = x^2$,

$$\int_0^{2\sqrt{2}} \frac{x}{2\sqrt{2}}\sqrt{1-\frac{x^2}{8}}\,\mathrm{d}x = \frac{1}{4\sqrt{2}}\int_0^8 \sqrt{1-\frac{u}{8}}\,\mathrm{d}u = \frac{1}{4\sqrt{2}}\,\frac{-2\cdot 8}{3}\left(1-\frac{u}{8}\right)^{\frac{3}{2}}\bigg|_{u=0}^{8} = \frac{2\sqrt{2}}{3}\ .$$

$\boxed{2\text{ pts}}$: 不定積分計算正確, $\boxed{1\text{ pt}}$: 求值正確