1072微甲及模組班期中考補考解答和評分標準

1. (11 pts) Consider a space curve $\vec{r}(t) = (\sin t, \sqrt{3}\sin t, -\cos t + 1)$.
   (a) (5 pts) Find the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$.
   (b) (6 pts) Find the maximum and minimum value of the curvature.

Solution:

(a) $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$, $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$. **(1 pt.)**

$\vec{r}'(t) = (\cos t, \sqrt{3}\cos t, \sin t)$ **(0.5 pt.)** and $|\vec{r}'(t)| = \sqrt{1+3\cos^2 t}$ **(0.5 pt.)**

Hence,

(3 pts.)
$$\begin{cases} \vec{T}(t) &= \dfrac{1}{|\vec{r}'(t)|}\vec{r}'(t) = \dfrac{1}{\sqrt{1+3\cos^2 t}}(\cos t, \sqrt{3}\cos t, \sin t), \\ \vec{T}'(t) &= \dfrac{1}{(1+3\cos^2 t)^{\frac{3}{2}}}(-\sin t, -\sqrt{3}\sin t, 4\cos t), \\ |\vec{T}'(t)| &= \dfrac{2}{1+3\cos^2 t}, \\ \vec{N}(t) &= \dfrac{1}{2\sqrt{1+3\cos^2 t}}(-\sin t, -\sqrt{3}\sin t, 4\cos t). \end{cases}$$

(b) $\kappa(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ **(1 pt.)**

(2 pts.)
$$\begin{cases} \vec{r}' \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sqrt{3}\cos t & \sin t \\ -\sin t & -\sqrt{3}\sin t & \cos t \end{vmatrix} = (\sqrt{3}, -1, 0), \\ |\vec{r}' \times \vec{r}''(t)| &= 2, \\ \kappa(t) &= \dfrac{2}{(1+3\cos^2 t)^{3/2}}. \end{cases}$$

$\because 0 \le \cos^2 t \le 1 \therefore 1 \le (1+3\cos^2 t)^{\frac{3}{2}} \le 8$.

Hence $\dfrac{1}{4} \le \kappa(t) \le 2$,

$$\begin{cases} \kappa(t) = 2 \text{ when } t = \dfrac{\pi}{2} + n\pi, \text{ for all } n \in \mathbb{N}. \\ \kappa(t) = \dfrac{1}{4} \text{ when } t = n\pi, \text{ for all } n \in \mathbb{N}. \end{cases}$$

Therefore, the maximum value of the curvature is 2. The minimum value of the curvature is $\dfrac{1}{4}$.

(3 pts.)

Solution 2: $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$ **(1 pt.)**, then $\kappa(t) = \dfrac{2}{(1+3\cos^2 t)^{3/2}}$ **(2 pts.)**.

2. (12 pts) Let $f(x,y) = \begin{cases} \dfrac{\sin(x^2y)}{x^4+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.

(a) (6 pts) Compute f_x and f_y for all (x,y) including $(0,0)$.

(b) (6 pts) Is $f(x,y)$ continuous at $(0,0)$? Is $f(x,y)$ differentiable at $(0,0)$? Justify your answers.

Solution:

(a) For $(x,y) \neq (0,0)$,

$$
\begin{aligned}
f_x(x,y) &= \frac{2xy\cos(x^2y)}{x^4+y^2} - \frac{4x^3\sin(x^2y)}{(x^4+y^2)^2} && \textbf{(2 pts.)} \\
f_y(x,y) &= \frac{x^2\cos(x^2y)}{x^4+y^2} - \frac{2y\sin(x^2y)}{(x^4+y^2)^2} && \textbf{(2 pts.)} \\
f_x(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0 && \textbf{(1 pt.)} \\
f_y(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0 && \textbf{(1 pt.)}
\end{aligned}
$$

(b) $f(x,0) = \dfrac{\sin 0}{x^4} = 0$ for all $x \neq 0$.

Hence $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis.

However, on the curve $y = x^2$, $f(x,x^2) = \dfrac{\sin(x^4)}{x^4+x^4} = \dfrac{1}{2}\dfrac{\sin(x^4)}{x^4} \to \dfrac{1}{2}$ as $x \to 0$.

Hence, $f(x,y) \to \dfrac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $y = x^2$.

Because $f(x,y)$ approaches different limits as (x,y) approaches $(0,0)$ along different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

Then $f(x,y)$ is not continuous at $(0,0)$.

(4 pts.)

Because differentiability implies continuity, we conclude that f is not differentiable at $(0,0)$.

(2 pts.)

3. (12 pts) Let $f(x,y) = xg(\frac{y}{x})$, where g is a differentiable function with $g(1) = -1$, $g'(1) = 2$.

(a) (4 pts) Use linear approximation to estimate the value of $f(2.01, 1.98)$.

(b) (4 pts) Suppose that at $(x,y) = (2,2)$, $g(\frac{y}{x})$ decreases most rapidly in the direction $\vec{u}$, where $|\vec{u}| = 1$. Find $D_{\vec{u}} f(2,2)$.

(c) (4 pts) If near the point $(2,2,-2)$, the surface $z = f(x,y)$ defines x implicitly as a function of y and z, $x = h(y,z)$. Find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ when $(y,z) = (2,-2)$.

Solution:

(a) The linear approximation of $f(x,y)$ at $2,2$ is

$$f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2) \qquad \textbf{(1 pt.)}$$

$$\begin{aligned} f(2,2) &= 2g\left(\frac{2}{2}\right) = 2 \cdot g(1) = -2 \\ f_x(x,y) &= g\left(\frac{y}{x}\right) - x \cdot \frac{y}{x^2} g'\left(\frac{y}{x}\right) = g\left(\frac{y}{x}\right) - \frac{y}{x} g'\left(\frac{y}{x}\right) \\ f_x(2,2) &= g(1) - g'(1) = -3 \\ f_y(x,y) &= g'\left(\frac{y}{x}\right), \quad f_y(2,2) = g'(1) = 2 \end{aligned}$$

(2 pts.)

Hence

$$\begin{aligned} f(2.01, 1.98) &\approx f(2,2) + f_x(2,2)(2.01-2) + f_y(2,2)(1.98-2) \\ &= -2 - 3 \times 0.01 + 2 \times (-0.02) = -2.07 \end{aligned}$$

(1 pt.)

(b) Let $h(x,y) = g\left(\frac{y}{x}\right)$, $\vec{u} = -\frac{\vec{\nabla} h}{|\vec{\nabla} h|}(2,2)$

$$\vec{\nabla} h(x,y) = \left(-\frac{y}{x^2} g'\left(\frac{y}{x}\right), \frac{1}{x} g'\left(\frac{y}{x}\right)\right) \quad // \quad (-y, x)$$

Hence $\vec{u} = -\frac{\vec{\nabla} h(2,2)}{|\vec{\nabla} h(2,2)|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

(2 pts.)

Because $\frac{y}{x}$ is differentiable at $(x,y) = (2,2)$ and g is differentiable, we know that $f(x,y) = x \cdot g\left(\frac{y}{x}\right)$ is differentiable at $(x,y) = (2,2)$. Therefore,

$$D_{\vec{u}} f(2,2) \overset{\textbf{(1 pt.)}}{=} \vec{\nabla} f(2,2) \cdot \vec{u} = (-3, 2) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \overset{\textbf{(1 pt.)}}{=} -\frac{5}{\sqrt{2}}.$$

(c) $z = x \cdot g\left(\frac{y}{x}\right) \Leftrightarrow F(x,y,z) = 0$ where $F(x,y,z) = x \cdot g\left(\frac{y}{x}\right) - z$.
Near the point $(2,2,-2)$, the level surface $F(x,y,z) = 0$ defines x implicitly as a function of y and z.
And

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}. \qquad \textbf{(2 pts.)}$$

At $(2, 2, -2)$,

$$\begin{aligned} F_x(2,2,-2) &= f_x(2,2) = -3 \\ F_y(2,2,-2) &= f_y(2,2) = 2 \\ F_z(2,2,-2) &= -1 \end{aligned}$$

Hence

$$\frac{\partial x}{\partial y} = \frac{2}{3}, \quad \frac{\partial x}{\partial z} = -\frac{1}{3}. \qquad \textbf{(2 pts.)}$$

4. (12 pts) Suppose that $(\sqrt{2}, \sqrt{2})$ is a critical point of $f(x,y) = x^3 + \alpha x^2 y + y^3 + \beta y$, where α, β are constants.

(a) (2 pts) Find values of α and β.

(b) (10 pts) Find and classify all critical points of $f(x,y)$.

Solution:

(a) $(\sqrt{2}, \sqrt{2})$ is a critical point of $f(x,y)$ (f is differentiable), then

$$f_x(\sqrt{2}, \sqrt{2}) = 0, \quad f_y(\sqrt{2}, \sqrt{2}) = 0. \qquad \textbf{(1 pt.)}$$

$$\begin{cases} f_x(\sqrt{2}, \sqrt{2}) = 3x^2 + 2\alpha xy\Big|_{(x,y)=(\sqrt{2},\sqrt{2})} = 6 + 4\alpha = 0 \\ f_y(\sqrt{2}, \sqrt{2}) = \alpha x^2 + 3y^2 + \beta\Big|_{(x,y)=(\sqrt{2},\sqrt{2})} = 6 + 2\alpha + \beta = 0 \end{cases}$$

Hence

$$\alpha = -\frac{3}{2}, \quad \beta = -3. \qquad \textbf{(1 pt.)}$$

(b) $f(x,y) = x^3 - \frac{3}{2}x^2 y + y^3 - 3y$.

Solve

$$\begin{cases} f_x(x,y) = 3x^2 - 3xy = 0 \\ f_y(x,y) = -\frac{3}{2}x^2 + 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{cases} x(x-y) = 0 \\ x^2 - 2y^2 + 2 = 0 \end{cases}$$

Hence

$$(x,y) = (0, \pm 1) \quad \text{or} \quad (x,y) = (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}). \qquad \textbf{(2 pts.)}$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x - 3y & -3x \\ -3x & 6y \end{vmatrix}$$

$$D(0,1) = \begin{vmatrix} -3 & 0 \\ 0 & 6 \end{vmatrix} < 0 \Rightarrow (0,1) \text{ is a saddle point} \qquad \textbf{(2 pts.)}$$

$$D(0,-1) = \begin{vmatrix} 3 & 0 \\ 0 & -6 \end{vmatrix} < 0 \Rightarrow (0,-1) \text{ is a saddle point} \qquad \textbf{(2 pts.)}$$

$$D(\sqrt{2}, \sqrt{2}) = \begin{vmatrix} 3\sqrt{2} & -3\sqrt{2} \\ -3\sqrt{2} & 6\sqrt{2} \end{vmatrix} = 18 > 0 \text{ and } f_{xx}(\sqrt{2}, \sqrt{2}) = 3\sqrt{2} > 0$$

$$\Rightarrow f(\sqrt{2}, \sqrt{2}) \text{ is a local minimum value.} \qquad \textbf{(2 pts.)}$$

$$D(-\sqrt{2}, -\sqrt{2}) = \begin{vmatrix} -3\sqrt{2} & 3\sqrt{2} \\ 3\sqrt{2} & -6\sqrt{2} \end{vmatrix} = 18 > 0 \text{ and } f_{xx}(-\sqrt{2}, -\sqrt{2}) = -3\sqrt{2} < 0$$

$$\Rightarrow f(-\sqrt{2}, -\sqrt{2}) \text{ is a local maximum value.} \qquad \textbf{(2 pts.)}$$

5. (15 pts) (a) (8 pts) Find the shortest distance between the point $(0,0,1)$ and the surface $y^2 = x^2 + 2z^2 + 1$.

(b) (7 pts) Let curve C be the intersection of the surface $y^2 = x^2+2z^2+1$ and the sphere $x^2+y^2+z^2 = 2$. Find the points on the curve C which are respectively the closest to and the farthest from the point $(0,0,1)$.

Solution:

(a) The square of the distance between $(0,0,1)$ and (x,y,z) is $f(x,y,z) = x^2 + y^2 + (z-1)^2$. Under the constraint $g(x,y,z) = x^2 - y^2 + 2z^2 + 1 = 0$, we want to find the minimum value of $f(x,y,z)$.

By Lagrange's multiplier method, we solve the equations:

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_x \Rightarrow 2x = \lambda(2x) & ---(1) \\ f_y = \lambda g_y \Rightarrow 2y = \lambda(-2y) & ---(2) \\ f_z = \lambda g_z \Rightarrow 2(z-1) = \lambda(4z) & ---(3) \\ x^2 - y^2 + 2z^2 + 1 = 0 & ---(4) \end{cases}$$

(3 pts.)

$(1) \Rightarrow (1-\lambda)x = 0 \Rightarrow x = 0$ or $\lambda = 1$.

If $x = 0$ $(2) \Rightarrow (1+\lambda)y = 0 \Rightarrow y = 0$ or $\lambda = -1$.

Case 1: $y = 0$ i.e. $x = y = 0$, (4) cannot be satisfied $\Rightarrow$ no solution.

Case 2: $\lambda = -1$, $(3) \Rightarrow z = \frac{1}{3}$, $(4) \Rightarrow y^2 = \frac{11}{9}$, $y = \pm\frac{\sqrt{11}}{3} \Rightarrow (x,y,z) = \left(0, \pm\frac{\sqrt{11}}{3}, \frac{1}{3}\right)$.

If $\lambda = 1$ $(2) \Rightarrow y = 0$, (4) cannot be satisfied $\Rightarrow$ no solution.

Hence the extreme value of $f(x,y,z)$ may occur at $\left(0, \frac{\sqrt{11}}{3}, \frac{1}{3}\right)$ or $\left(0, -\frac{\sqrt{11}}{3}, \frac{1}{3}\right)$

(4 pts.)

$f\left(0, \pm\frac{\sqrt{11}}{3}, \frac{1}{3}\right) = \frac{5}{3}$ and this should be the minimum value ($\because$ the surface $g(x,y,z) = 0$ is unbounded $\therefore$ $f(x,y,z)$ has no upper bound on $g = 0$.)

Ans: the distance between $(0,0,1)$ and surface $g(x,y,z) = 0$ is $\sqrt{\frac{5}{3}}$.

(b) We want to find the extreme values of $f(x,y,z) = x^2 + y^2 + (z-1)^2$ under constraints $g_1(x,y,z) = x^2 - y^2 + 2z^2 + 1 = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 = 2$.
By Lagrange multiplier method, we solve the equations:

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g_1 + \mu \vec{\nabla} g_2 \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_{1x} + \mu g_{2x} \\ f_y = \lambda g_{1y} + \mu g_{2y} \\ f_z = \lambda g_{1z} + \mu g_{2z} \\ x^2 - y^2 + 2z^2 + 1 = 0 \\ x^2 + y^2 + z^2 = 2 \end{cases} \Rightarrow \begin{cases} 2x = \lambda(2x) + \mu(2x) \\ 2y = \lambda(-2y) + \mu(2y) \\ 2(z-1) = \lambda(4z) + \mu(2z) \\ x^2 - y^2 + 2z^2 + 1 = 0 \\ x^2 + y^2 + z^2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} (1-\lambda-\mu)x = 0 & ---(1) \\ (1+\lambda-\mu)y = 0 & ---(2) \\ (1-2\lambda-\mu)z = 1 & ---(3) \\ x^2 - y^2 + 2z^2 + 1 = 0 & ---(4) \\ x^2 + y^2 + z^2 = 2 & ---(5) \end{cases}$$

(3 pts.)

(1) $\Rightarrow x = 0$ or $1 = \lambda + \mu$.

If $x = 0$, (4) and (5) $\Rightarrow z^2 = \frac{1}{3}$, $y^2 = \frac{5}{3} \Rightarrow$ solutions are $(x,y,z) = \left(0, \sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right), \left(0, -\sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(0, \sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$, or $\left(0, -\sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$.

If $1 = \lambda + \mu$, (2) $\Rightarrow 1 + \lambda - \mu = 0$ or $y = 0$.

Case 1: $1 + \lambda - \mu = 0 \Rightarrow \lambda = 0, \mu = 1$. (3) is not satisfied $\Rightarrow$ no solutions.

Case 2: $y = 0$, (4) cannot be satisfied $\Rightarrow$ no solution.

Hence the extreme values may occur at $(x,y,z) = \left(0, \sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right), \left(0, -\sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right), \left(0, \sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$, or $\left(0, -\sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$.

$$f\left(0, \pm\sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right) = 3 - \frac{2}{\sqrt{3}}, \quad f\left(0, \pm\sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right) = 3 + \frac{2}{\sqrt{3}}.$$

(3 pts.)

Ans: $\left(0, \sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(0, -\sqrt{\frac{5}{3}}, \frac{1}{\sqrt{3}}\right)$ are closest to the point $(0,0,1)$.
$\left(0, \sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(0, -\sqrt{\frac{5}{3}}, -\frac{1}{\sqrt{3}}\right)$ are farthest from the point $(0,0,1)$.

(1 pt.)

6. (12 pts) Evaluate the double integral

(a) (5 pts) $\int_0^2 \int_0^{4-x^2} \frac{x\, e^{2y}}{4-y}\, dy\, dx$.

(b) (7 pts) $\int\int_{\mathcal{R}} x^2\, dA,$, where $\mathcal{R}$ is the region bounded by the ellipse $(x-y)^2 + 2y^2 = 1$.

Solution:

(a)

$$\begin{aligned} & \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx \\ = & \int_{y=0}^4 \int_{x=0}^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dxdy \quad (\mathbf{2pts}.) \\ = & \int_{y=0}^4 \frac{e^{2y}}{4-y} \left(\frac{x^2}{2}\right)\Big|_0^{\sqrt{4-y}} dy \quad (\mathbf{1pt}.) \\ = & \int_{y=0}^4 \frac{1}{2} e^{2y} dy \quad (\mathbf{1pt}.) \\ = & \frac{1}{4} e^{2y}\Big|_0^4 \\ = & \frac{1}{4}(e^8 - 1) \quad (\mathbf{1pt}.) \end{aligned}$$

(b)

$$\int\int_R x^2 dA, \quad R : (x-y)^2 + 2y^2 \le 1.$$

Let

$$\begin{aligned} u &= x - y \\ v &= \sqrt{2}y \end{aligned} \Rightarrow \begin{aligned} x &= u + y = u + \frac{v}{\sqrt{2}} \\ y &= \frac{v}{\sqrt{2}} \end{aligned} \quad (\mathbf{2pts}.)$$

We have

$$R' : u^2 + v^2 \le 1 \quad \& \quad \int\int_{R'} \left(u + \frac{v}{\sqrt{2}}\right)^2 |J| dudv \quad (\mathbf{1pt}.)$$

where $|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}$.

Now let

$$\begin{aligned} u &= r\cos\theta \\ v &= r\sin\theta \end{aligned}$$

We find

$$R'' : r^2 \le 1 \quad (\mathbf{2pts}.)$$

and

$$\begin{aligned} & \int\int_{R''} \frac{1}{\sqrt{2}} \left(r^2\cos^2\theta + \frac{2}{\sqrt{2}} r^2 \cos\theta\sin\theta + \frac{1}{2} r^2 \sin^2\theta\right) rdrd\theta \\ = & \int_{\theta=0}^{2\pi} \int_{r=0}^1 \frac{1}{\sqrt{2}} r^3 \left[\frac{1}{2}(1+\cos\theta) + \frac{1}{\sqrt{2}}\sin(2\theta) + \frac{1}{4}(1-\cos(2\theta)) drd\theta\right] \\ = & \frac{3\pi}{8\sqrt{2}} \quad (\mathbf{2pts}.) \end{aligned}$$

7. (14 pts) (a) (7 pts) Find the volume of the wedge cut out of the cylinder $x^2+y^2=1$, $z\geq 0$ by the plane $z=-y$.

(b) (7 pts) Find the volume of the region cut out of the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=3y$.

Solution:

(a)

$$A(x)=\frac{1}{2}y^2=\frac{1}{2}(1-x^2) \qquad \textbf{(3pts.)}$$

$$\begin{aligned} V &= \int_{-1}^{1} A(x)dx \\ &= \int_{-1}^{1}\frac{1}{2}(1-x^2)dx \quad \textbf{(2pts.)} \\ &= \frac{1}{2}\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} \\ &= \frac{2}{3} \qquad \textbf{(2pts.)} \end{aligned}$$

(b) The cylinder

$$x^2+y^2=3y$$

written in polar coordinate is

$$r^2=3r\sin\theta \Rightarrow r=2\sin\theta \qquad \textbf{(1pt.)}$$

The sphere $x^2+y^2+z^2=9$ written in cylindrical coordinate is

$$r^2+z^2=9. \qquad \textbf{(1pt.)}$$

The integral over cylindrical region is

$$\begin{aligned} V_c &= \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta \qquad \textbf{(1pt.)} \\ &= \int_{\theta=0}^{\pi}\int_{r=0}^{3\sin\theta} r\sqrt{9-r^2}\,dr\,d\theta \qquad \textbf{(1pt.)} \\ &= \int_{\theta=0}^{\pi} -\frac{1}{3}\cdot 9^{3/2}\left[(1-\sin^2\theta)^{3/2}-1\right]d\theta \\ &= \int_{\theta=0}^{\pi} -9(\cos^3\theta-1)d\theta \qquad \textbf{(1pt.)} \\ &= \int_{\theta=0}^{\pi} -9\left[(1-\sin^2\theta)\cos\theta-1\right]d\theta \\ &= 9\pi. \qquad \textbf{(1pt.)} \end{aligned}$$

Thus the volume of the region of the sphere cut by the cylinder is

$$V=\frac{4}{3}\pi\cdot 3^3-2\cdot 9\pi=18\pi. \qquad \textbf{(1pt.)}.$$

8. (12 pts) Find the center of mass of a thin plate occupying the smaller region cut out of the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ with the density $\rho = 5x$.

Solution:

The intersection points of $x^2 + 4y^2 = 12$ and $x = 4y^2$ are

$$x^2 + x = 12 \Rightarrow x = -4(\times) \text{ or } x = 3. \qquad (\mathbf{2pts}.)$$

The mass over region R is

$$\begin{aligned} m &= \int\int_R \rho dA \\ &= \int_{y=0}^{\sqrt{3/4}} \int_{x=4y^2}^{(12-4y^2)^{1/2}} 5x dx dy \qquad (\mathbf{1pt}.) \\ &= 2\int_0^{\sqrt{3/4}} \frac{5}{2}(12 - 4y^2 - 16y^4) dy \quad (\mathbf{1pt}.) \\ &= 5\left(12y - \frac{4}{3}y^3 - \frac{16y^5}{5}\right)\Big|_0^{\sqrt{3/4}} \\ &= 23\sqrt{3} \qquad (\mathbf{1pt}.) \end{aligned}$$

$\bar{x} = \dfrac{\int\int_R x\rho dA}{m} = \dfrac{M_y}{m}$ where

$$\begin{aligned} M_y &= 2\int_0^{\sqrt{3/4}} \int_{4y^2}^{(12-4y^2)^{1/2}} 5x^2 dx dy \\ &= 2\int_0^{\sqrt{3/4}} \frac{5}{3}\left[(12 - 4y^2)^{3/2} - (4y^2)^3\right] dy \quad (\mathbf{3pts}.) \\ &= 15\pi + \frac{765}{28}\sqrt{3}. \qquad (\mathbf{2pts}.) \end{aligned}$$

and $\bar{y} = 0$. **(2 pts.)**