1071微甲及模組班期中考解答和評分標準

1. (15 pts) Find the following limits.

(a) (5 pts) $\lim_{x\to 0} \sqrt[3]{x}\sin\left(\sin\left(\frac{1}{|x|}\right)\right)$.

(b) (5 pts) $\lim_{x\to \infty} \sqrt[3]{x}\sin\left(\sin\left(\frac{1}{|x|}\right)\right)$.

(c) (5 pts) $\lim_{x\to -\infty} \left(\sqrt{x^2+2x-1}+x\right)$.

Solution:

(a) $-1 \le \sin(\left(\sin\frac{1}{|x|}\right)) \le 1$ (2 pts)

$-|\sqrt[3]{x}| \le \sqrt[3]{x}\sin(\left(\sin\frac{1}{|x|}\right)) \le |\sqrt[3]{x}|$ (1 pt)

$\lim_{x\to 0}|\sqrt[3]{x}| = \lim_{x\to 0} -|\sqrt[3]{x}| = 0$ (1 pt)

By squeeze principle,

$\lim_{x\to 0}|\sqrt[3]{x}|\sin(\sin(\frac{1}{|x|})) = 0$ (1 pt)

(b) Method 1(L'Hospital's rule):

$\lim_{x\to\infty} \sqrt[3]{x}\sin(\sin(\frac{1}{|x|}))$ is a $\infty\cdot 0$ indeterminate form, so we need to rewrite it to the 0/0 indeterminate form as follow:

$$\lim_{x\to\infty} \sqrt[3]{x}\sin(\sin(\frac{1}{|x|})) = \lim_{x\to\infty} \frac{\sin(\sin(\frac{1}{|x|}))}{x^{-\frac{1}{3}}} \qquad 1'$$

Using L'Hospital's rule, it follows that:

$$\lim_{x\to\infty} \frac{\sin(\sin(\frac{1}{|x|}))}{x^{-\frac{1}{3}}} = \lim_{x\to\infty} \frac{\cos(\sin(\frac{1}{x}))\cdot\cos(\frac{1}{x})\cdot(-x^{-2})}{-\frac{1}{3}x^{-\frac{4}{3}}} \qquad 2'$$

(Notice that $x\to\infty$ implies $x>0$, so we can get rid of the tricky absolute value.)
After simplifying the formula, we get:

$$\lim_{x\to\infty} \frac{\cos(\sin(\frac{1}{x}))\cdot\cos(\frac{1}{x})\cdot(-x^{-2})}{-\frac{1}{3}x^{-\frac{4}{3}}} = \lim_{x\to\infty} \frac{\cos(\sin(\frac{1}{x}))\cdot\cos(\frac{1}{x})}{\frac{1}{3}x^{\frac{2}{3}}} \qquad 1'$$

(The second using of L'Hospital's rule is not necessary.)
Since the numerator approaches to 1 and the denominator approaches to ∞, we can make a conclusion that:

$$\lim_{x\to\infty} \sqrt[3]{x}\sin(\sin(\frac{1}{|x|})) = 0 \qquad 1'$$

Method 2($\lim_{x\to 0}\sin x/x = 1$):

$$\lim_{x\to\infty} \sqrt[3]{x}\sin(\sin(\frac{1}{|x|})) = \lim_{x\to\infty} \frac{\sqrt[3]{x}}{|x|}\cdot\frac{\sin(\sin(\frac{1}{|x|}))}{\sin(\frac{1}{|x|})}\cdot\frac{\sin(\frac{1}{|x|})}{\frac{1}{|x|}}$$

Since $\lim_{x\to\infty} \frac{\sqrt[3]{x}}{|x|} = 0$, $1'$

$$\lim_{x\to\infty} \frac{\sin(\sin(\frac{1}{|x|}))}{\sin(\frac{1}{|x|})} = \lim_{y\to 0} \frac{\sin(y)}{y} = 1, \qquad (\sin(\frac{1}{|x|}) = y) \qquad 1'$$

$$\lim_{x\to\infty} \frac{\sin(\frac{1}{|x|})}{\frac{1}{|x|}} = \lim_{x\to 0} \frac{\sin(y)}{y} = 1, \qquad (\frac{1}{|x|} = z) \qquad 1'$$

(You can get another $1'$ if the above three points is all correct.)

$$\lim_{x\to\infty} \sqrt[3]{x} \sin(\sin(\frac{1}{|x|})) = 0 \cdot 1 \cdot 1 = 0 \qquad 1'$$

(c)

$$\lim_{x\to-\infty} \sqrt{x^2+2x-1} + x = \lim_{x\to-\infty} \sqrt{x^2+2x-1} + x \cdot \frac{(\sqrt{x^2+2x-1} - x)}{(\sqrt{x^2+2x-1} - x)} \qquad 2\text{pt}$$

$$= \lim_{x\to-\infty} \frac{2x-1}{\sqrt{x^2+2x-1} - x} \text{(Let t=-x)}$$

$$= \lim_{t\to\infty} \frac{-2t-1}{\sqrt{t^2-2t-1} + t}$$

$$= \lim_{t\to\infty} \frac{-2-\frac{1}{t}}{\sqrt{1-\frac{2}{t}-\frac{1}{t^2}} + 1}$$

$$= -1 \qquad 3\text{pt}$$

2. (20 pts) Find the first-order derivative of the following functions.

(a) (5 pts) $f(x) = \dfrac{\tan^{-1} x}{1 + xe^x}$

(b) (5 pts) $f(x) = \ln\left(\dfrac{\sqrt{\sin x \cos x}}{1 + 2\ln x}\right)$

(c) (5 pts) $f(x) = x\tan(\sin^{-1} x)$

(d) (5 pts) $x^y = y^x + y$. Find $\dfrac{dy}{dx}$ at $(2, 1)$.

Solution:

(a) Let $g(x) = 1 + xe^x$.

Since $g'(x) = e^x(x+1) < 0$ on $(-\infty, -1)$,$g'(x) > 0$ on $(-1, \infty)$ and $g'(x)$ changes sign at -1,we know that g has absolute minimum at -1.

$\Rightarrow g(x) \geq g(-1) = 1 - e^{-1} > 0, \forall x \in \mathbb{R}$.(i.e. $1 + xe^x \neq 0$ for all $x \in \mathbb{R}$.)

Now since $1 + xe^x \neq 0$ on $\mathbb{R}$ and $\tan^{-1} x, 1 + xe^x$ are both differentiable on $\mathbb{R}$,we can apply Quotient Rule to $f(x)$ to conclude that

$$f'(x) = \frac{(1 + xe^x)\frac{d}{dx}(\tan^{-1} x) - \tan^{-1}\frac{d}{dx}(1 + xe^x)}{(1 + xe^x)^2}$$

$$= \frac{(1 + xe^x)(\frac{1}{1+x^2}) - (\tan^{-1} x)(e^x + xe^x)}{(1 + xe^x)^2}, \forall x \in \mathbb{R}.$$

$$(\text{Since } \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}, \frac{d}{dx}(1 + xe^x) = (e^x + xe^x))$$

評分標準:

1. 對 f 取 ln 微分(f 的 Range 有負數和零，不能亂取 ln)(-1分)
2. $\arctan x$ 微分微錯(-1分)
3. $1 + xe^x$ 微分微錯(-1分)

Marking guidelines:

1. applying logarithmic differentiation to f without absolute sign (-1 point)
2. mistake in differentiating $\arctan x$ (-1 point)
3. mistake in differentiating $1 + xe^x$ (-1 point)

(b) **[M1]**

$$f'(x) = \frac{1 + 2\ln x}{\sqrt{\sin x \cos x}} \cdot \frac{\frac{1}{2}\frac{1}{\sqrt{\sin x \cos x}}\left(\cos^2 x - \sin^2 x\right)(1 + 2\ln x) - \sqrt{\sin x \cos x} \cdot \frac{2}{x}}{(1 + 2\ln x)^2}$$

$$= \frac{x(1 + 2\ln x)\cos 2x - 4\sin x \cos x}{2x(1 + 2\ln x)\sin x \cos x}$$

[M2]

$$f(x) = \ln\left(\frac{\sqrt{\sin x \cos x}}{1 + 2\ln x}\right)$$

$$= \frac{1}{2}\ln(\sin x \cos x) - \ln(1 + 2\ln x)$$

Thus, $f'(x) = \dfrac{1}{2\sin x \cos x}\left(\cos^2 x - \sin^2 x\right) - \dfrac{2}{x(1 + 2\ln x)}$

Any mistake in chain rule or product rule, 1 place -2 pts. Any mistake in simplfy calculation, -1 pt most.

(c) 經過討論，此題應將三角與反三角的合成做化簡，因此分為法一與法二，在沒有化簡的法二中，滿分上限將會是4分。

(It is required that the composite of the trigonometric functions in the question has to be simplified to the form expressible as composition of rational functions and radicals. Two solutions are given below, in which the answer in the second one is not simplified and a maximum of 4 points can be scored for such solution.)

Solution 1:

令$y=\sin^{-1}x$，也就有$\sin y = x$，可將$\tan(\sin^{-1}x)$化簡為$\dfrac{x}{\sqrt{1-x^2}}$，故

$$f(x)=\frac{x^2}{\sqrt{1-x^2}}$$

由微分的除法公式以及連鎖規則可得到

$$\begin{aligned}f'(x)&=\frac{2x\sqrt{1-x^2}-x^2\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)}{1-x^2}\\&=\frac{2x(1-x^2)+x^3}{(1-x^2)\sqrt{1-x^2}}\\&=\frac{2x-x^3}{(1-x^2)^{3/2}}\end{aligned}$$

註:

1. 此方法滿分為5分
2. 有寫出化簡過後的$f(x)$則至少有1分
3. chain rule或是除法規則出錯都會被各扣1分
4. 若寫到微分式子的第一個等號而沒有往下作化簡，會被扣1分
5. 在化簡過程中出現錯誤而導致答案出錯，最多仍可能獲得4分

Remarks:

1. The full score for solutions using this method is 5 points.
2. Expressing $f(x)$ in its simplified form scores at least 1 point.
3. 1 point is deducted for mistakes in applying the chain rule, and 1 point for those in applying the quotient rule.
4. 1 point is deducted if the derivative is not simplified after applying the chain rule and/or the quotient rule.
5. A maximum of 4 points can still be scored if there are mistakes in the simplification process.

Solution 2:

由於$(\tan x)'=\sec^2 x$且$(\sin^{-1}x)'=\dfrac{1}{\sqrt{1-x^2}}$，故由微分的乘法公式以及連鎖規則可得到

$$f'(x)=\tan(\sin^{-1}x)+x\sec^2(\sin^{-1}x)\cdot\frac{1}{\sqrt{1-x^2}}$$

註:

1. 兩個函數的微分、chain rule、product rule都是各佔1分
2. 有許多同學寫出了上式之後試圖化簡卻「因計算出錯」而失敗，仍會獲得原本的4 分
3. 當你使用法二的時候是看不出你對於反三角概念的熟悉度，但有些同學會將「$\sin^{-1} x$」錯誤的化簡成「$\frac{1}{\sqrt{1-x^2}}$」，出現這樣的情況時，即使你前面寫出了正確式子，但在後面錯誤的化簡中顯露了錯誤的概念，將會被扣1至2 分

Remarks:

1. Derivatives of the two functions, the chain rule, and the product rule each worths 1 point.
2. A maximum of 4 points can still be scored with mistakes in the attempt to simplify the above derivative.
3. However, since this solution does not reveal your understanding of (inverse) trigonometric functions, if the mistakes in simplifying the above derivative are due to your misconception towards inverse trigonometric functions, 1 2 points are still deductible.

(d) The original equation can be cast as

$$e^{y\ln x} = e^{x\ln y} + y$$

Differentiating on both sides, we have

$$e^{y\ln x}\left(\frac{y}{x} + y'\ln x\right) = e^{x\ln y}\left(\frac{xy'}{y} + \ln y\right) + y'$$

Substituting $(2,1)$ into the equation gives

$$2\cdot\left(\frac{1}{2} + y'(2)\cdot\ln 2\right) = 1\cdot\left(\frac{2\cdot y'(2)}{1} + 0\right) + y'(2).$$

Or,

$$y'(2) = \frac{1}{3-2\ln 2}$$

2 points if the student carries out the calculation of the derivative, even with minor mistakes. 4 points if the derivative is correct and he or she substitutes $(2,1)$ into the expression and arranges, in addition to the above, even with minor mistakes. 5 points if the answer is correct, in addition to the above.

3. (15 pts) Let $f(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & \text{for } x \neq 0,\ x > -1, \\ a, & \text{for } x = 0. \end{cases}$

(a) (3 pts) Find the value of a such that $f(x)$ is continuous at $x = 0$.

(b) (4 pts) Find $\lim_{x\to\infty} f(x)$.

(c) (4 pts) Compute $f'(x)$, for $x \neq 0$.

(d) (4 pts) Is $f(x)$ differentiable at $x = 0$? If $f(x)$ is differentiable at $x = 0$, then find $f'(0)$.

Solution:

(a) [Method 1]

$$f(x) = (1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}.$$

Since both of x and $\ln(1+x)$ converge to 0 as $x \to 0$, (check 0/0 1pt) we may apply L'Hospital's rule: (1pt)

$$\lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1 \Rightarrow \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

(in other notations, $\lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{L}{=} \lim_{x\to 0} \frac{1}{1+x} = 1$)

Hence $\lim_{x\to 0} f(x) = e^1 = e$ by continuity of the exponential functions. Thus $a = e$ (1pt).

[Method 2] to calculate the limit $\lim_{x\to 0} f(x)$: for $x > 0$, by the definition of e:

$$\lim_{x\to 0+} f(x) = \lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t = e. \quad \text{(1pt)}$$

Also,

$$\begin{aligned}\lim_{x\to 0-} f(x) &= \lim_{t\to\infty} \left(1 - \frac{1}{t}\right)^{-t} = \lim_{t\to\infty} \left(\frac{t-1}{t}\right)^{-t} \\ &= \lim_{t\to\infty} \left(1 + \frac{1}{t-1}\right)^{t-1} \left(1 + \frac{1}{t-1}\right) = e \cdot 1 = e. \quad \text{(1pt)}\end{aligned}$$

[Method 3] another way to calculate the limit $\lim_{x\to 0} f(x)$:

$$f(x) = e^{\frac{1}{x}\ln(1+x)}.$$

$$\begin{aligned}\lim_{x\to 0} \frac{\ln(1+x)}{x} &= \lim_{x\to 0} \frac{\ln(1+x) - \ln 1}{x - 0} = \frac{d}{dx} \ln(1+x)\Big|_{x=0} \\ &= \frac{1}{1+x}\Big|_{x=0} = 1. \quad \text{(2pt)}\end{aligned}$$

Hence $\lim_{x\to 0} f(x) = e$.

(b)

$$f(x) = e^{\frac{1}{x}\ln(1+x)}.$$

Since both of x and $\ln(1+x)$ go to ∞ as $x \to \infty$ (check ∞/∞ 1pt), we may apply the L'Hospital's rule (1pt):

$$\lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0 \Rightarrow \lim_{x\to\infty} \frac{\ln(1+x)}{x} = 0$$

(may write $\lim_{x\to\infty} \frac{\ln(1+x)}{x} \overset{L}{=} \lim_{x\to\infty} \frac{1}{1+x} = 0$).

Hence $\lim_{x\to\infty} f(x) = e^0 = 1$ (2pt) by continuity of the exponential function.

Note: calculation mistakes, forget to write the notation "lim" → -1pt.

(c) method 1(using chain rule):

$$f(x) = e^{\frac{1}{x}\ln(1+x)} \tag{1}$$

Let $u = \frac{1}{x}\ln(1+x)$, and $f(x) = e^u$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} \tag{2}$$

(1pt)

$$\frac{df}{du} = \frac{de^u}{u} = e^u = e^{\frac{1}{x}\ln(1+x)} \tag{3}$$

(1pt)

$$\frac{du}{dx} = \frac{d(x^{-1}) \cdot \ln(1+x)}{x} = -x^{-2}\ln(1+x) + x^{-1}\frac{1}{1+x} \tag{4}$$

(2pt)

method 2(using implicit function theorem):

$$f(x) = e^{\frac{1}{x}\ln(1+x)} \tag{1}$$

$$\ln f(x) = \frac{1}{x}\ln(1+x) \tag{2}$$

(1pt)

$$\frac{d\ln f(x)}{x} = \frac{d\ln f(x)}{df(x)} \frac{df(x)}{dx} = \frac{f'(x)}{f(x)} \tag{3}$$

$$\frac{d\ln f(x)}{x} = \frac{d(x^{-1}\ln(1+x))}{dx} = -x^{-2}\ln(1+x) + x^{-1}\frac{1}{1+x} \tag{4}$$

(2pt)

Since (3) = (4) = $\frac{d\ln f(x)}{dx}$, we have $f'(x) = f(x) \cdot (4)$, so

$$f'(x) = f(x) \cdot (-x^{-2}\ln(1+x) + x^{-1}\frac{1}{1+x}) \tag{5}$$

$$= e^{\frac{1}{x}\ln(1+x)}(-x^{-2}\ln(1+x) + x^{-1}\frac{1}{1+x})$$

(1pt)

Some grading details:

- you won't get any point if you only apply the formula of differentiating polynomial (i.e. $f'(x) = \frac{1}{x}(1+x)^{\frac{1}{x}-1}$)

- you will get 1 point at part four if you have exactly one calculation mistake (i.e. $\frac{du}{dx} = -x^{-2}\ln(1+x)\cdot x^{-1}\frac{1}{1+x}$, or $-x^{-4}\ln(1+x)+x^{-3}\frac{1}{1+x}$, ...), and 0 point for more mistake.
- If you write $f(x) = e^{\frac{1+x}{x}}$ or something wrong slightly, and the calculation is reasonable, you will get one or two points in total.

(d)

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} \quad \left(\frac{0}{0}\right) \quad (1\%)$$

$$\xrightarrow{L'H} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}\cdot\left(\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right)}{1} \quad by(c) \quad (1\%)$$

$$= e\cdot\lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2(1+x)} \quad \left(\frac{0}{0}\right)$$

$$\xrightarrow{L'H} = e\cdot\lim_{x\to 0}\frac{1-\ln(1+x)-(1+x)\cdot\frac{1}{1+x}}{3x^2+2x}$$

$$= e\cdot\lim_{x\to 0}\frac{-\ln(1+x)}{3x^2+2x} \quad \left(\frac{0}{0}\right)$$

$$\xrightarrow{L'H} = e\cdot\lim_{x\to 0}\frac{-\frac{1}{1+x}}{6x+2}$$

$$= -\frac{1}{2} \quad (1\%)$$

$$\Rightarrow f \text{ is differentiable at } x=0 \quad (1\%)$$

4. (12 pts)

(a) (4 pts) Find the linearization of $f(x) = \sin^{-1} x$ at $x = 0.5$. Denote the linearization by $L(x)$.

(b) (4 pts) Use linear approximation to estimate $\sin^{-1}(0.49)$.

(c) (4 pts) Let $g(x) = \sin^{-1} x - L(x)$. Use the Mean Value Theorem twice to estimate $|g(0.49) - g(0.5)|$ and get an upper bound for the quantity.

Solution:

(a) The linearization of $f(x) = \sin^{-1} x$ at $x = 0.5$ is $L(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2})$. **(2分)**

$f(\frac{1}{2}) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ **(1分)**

Since $f'(x) = \frac{1}{\sqrt{1-x^2}}$, one has $f'(\frac{1}{2}) = \frac{2}{\sqrt{3}}$. **(1分)**

Therefore, $L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \frac{1}{2})$.

(b) The linear approximation of $\sin^{-1}$ at 0.5 reads

$$\sin^{-1} x \approx L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - 0.5) \ .$$

Therefore,

$$\sin^{-1}(0.49) \approx L(0.49) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(-0.01) = \frac{\pi}{6} - \frac{2}{\sqrt{3}}(0.01) \ .$$

(showing knowledge of $\sin^{-1} x \approx L(x)$: 2 points)

(substituting 0.49 into the linear approximation + answer: 1+1 points)

(c) Let $f(x) = \arcsin(x)$. We have known that $f'(x) = \frac{1}{\sqrt{1-x^2}}$, and $L(x) = f(a) + f'(a)(x-a)$.
Let $x = 0.49, a = 0.5$ then $g(0.5) = f(0.5) - [f(0.5) + f'(0.5)(0.5 - 0.5)] = 0$.
By Mean Value Theorem, $\exists\ x_1 \in [0.49, 0.5]$ such that $f(0.49) - f(0.5) = (0.49 - 0.5)f'(x_1)$.
By Mean Value Theorem, $\exists\ x_2 \in [x_1, 0.5]$ such that $f'(x_1) - f'(0.5) = (x_1 - 0.5)f''(x_2)$.

$$\begin{aligned}
& |\ g(0.49) - g(0.5)\ | \\
= & |\ f(0.49) - [f(0.5) + f'(0.5)(0.49 - 0.5)]\ | \\
= & |\ (0.49 - 0.5)(f'(x_1) - f'(0.5))\ | && \text{, for some } x_1 \in [x, a] \\
= & |\ f''(x_2)(0.49 - 0.5)(x_1 - 0.5)\ | && \text{, for some } x_2 \in [x_1, a] \\
\le & \max_{x \in [0.49, 0.5]} |f''(x)|\ (0.01)^2
\end{aligned}$$

(The estimation: 1pt).

$$\arcsin''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}} \qquad (1pt)$$

Since $f''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}}$ is a increasing function (it's trivial since the numerator is increasing and the denominator is decreasing), so its maximum in $[0.49, 0.05]$ happens at $x = 0.5$. (1pt)
So finally we obtain the upper bound:

$$|g(0.49) - g(0.5)| \le \arcsin''(0.5)(0.01)^2 = \frac{1}{10000}\frac{4}{\sqrt{27}} \qquad \text{(1pt)}$$

The twice Mean Value Theorem estimation: 1pt

$\arcsin''(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$: 1pt

The maximum of $arcsin''(x)$ in the interval $[0.49, 0.5]$: 1pt

The final answer $\dfrac{1}{10000}\dfrac{4}{\sqrt{27}}$: 1pt.

[01-02班] Suppose that $f(x)$ is twice differentiable. Let $L(x)$ be the linearization of $f(x)$ at $x = a$.

Define $g(x) = \begin{cases} \dfrac{f(x) - L(x)}{x - a}, & \text{for } x \neq a. \\ 0, & \text{for } x = a. \end{cases}$

(a) (3 pts) Show that $g(x)$ is continuous at $x = a$.

(b) (5 pts) Show that $g(x)$ is differentiable at $x = a$ and compute $g'(a)$. Write down the linearization, $L_g(x)$, of $g(x)$ at $x = a$.

(c) (4 pts) Because $f(x) = L(x) + g(x)(x - a)$, we can use $L(x) + L_g(x)(x - a)$ as a better approximation of $f(x)$. Use this better approximation to estimate $\tan^{-1}(1.01)$.

Solution:

(a) The linearization of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. **(1分)**

To prove that $g(x)$ is continuous at a, it suffices to show that $\lim_{x\to a} g(x) = g(a)$. **(0.5分)**

[Method I]

$$\begin{aligned} \lim_{x\to a} g(x) &= \lim_{x\to a} \frac{f(x) - L(x)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - f'(a) \\ &= f'(a) - f'(a) \\ &= 0 \end{aligned}$$

[Method II]

$$\begin{aligned} \lim_{x\to a} g(x) &= \lim_{x\to a} \frac{f(x) - L(x)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - f'(a) \\ &\overset{L}{=} \lim_{x\to a} \frac{f'(x)}{1} - f'(a) \\ &= f'(a) - f'(a) \quad \Big(f \text{ is twice differentiable and hence} \lim_{x\to a} f'(x) = f(a)\Big) \\ &= 0 \end{aligned}$$

Since $\lim_{x\to a} g(x) = g(a) = 0$, we can conclude that $g(x)$ is continuous at $x = a$.

(以上運算極限過程共1.5分)

(b)

$$g'(a) \overset{\text{def.}}{=} \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a} \frac{\frac{f(x)-L(x)}{x-a} - 0}{x - a} = \lim_{x\to a} \frac{f(x) - (f(a) + f'(a)(x - a)) - 0}{(x - a)^2}$$

(definition and computation: 1 point)

By using L'Hospital's Rule

$$\overset{L'H.R.}{=} \lim_{x\to a} \frac{f'(x) - f'(a)}{2(x - a)}$$

(correct use L'Hospital's Rule: 1 point)

$$= \frac{1}{2} \lim_{x\to a} \frac{f'(x) - f'(a)}{(x - a)} = \frac{1}{2} f''(a) \ .$$

(computation and definition: 1 point)

Therefore,

$$g'(a) = \frac{f''(a)}{2} .$$

(answer of $g'(a)$: 1 point)

$$L_g(x) = g(a) + g'(a)(x-a) = \frac{f''(a)}{2} \cdot (x-a) .$$

(answer of $L_g(x)$): 1 point)

(c) Let $f(x) = tan^{-1}(x)$ and $a = 1.01$, we have $arctan'(x) = \frac{1}{1+x^2}$, $arctan''(x) = \frac{-2x}{(1+x^2)^2}$ and $g(x) = \frac{f(x)-L(x)}{x-a} = \frac{f(x)-f(a)}{x-a} + f'(a)$.

One should solve $L_g(x) = \frac{1}{2}f''(a)(x-a)$ in part (b), so take $x = 1.01$ then

$$\begin{aligned} & L(x) + L_g(x)(x-a) \ \ |_{x=1.01,\ a=1} \\ = \ & f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \ \ |_{x=1.01,\ a=1} \\ = \ & arctan(1) + arctan'(1)(0.01) + \frac{1}{2}arctan''(1)(0.01)^2 \\ = \ & \frac{\pi}{4} + 0.005 - 0.000025 \\ = \ & \frac{\pi}{4} + 0.004975 (= \frac{\pi}{4} + \frac{199}{40000}) \end{aligned}$$

$arctan'(x) = \frac{1}{1+x^2}$: 1 pt

$arctan''(x) = \frac{-2x}{(1+x^2)^2}$: 1 pt

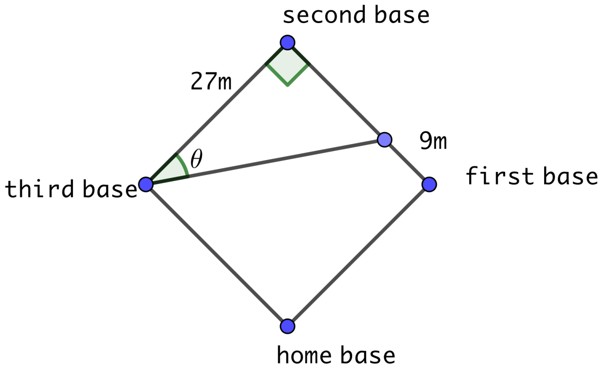
the final answer: 2 pts. (Wrong simplification just get 1 pt.)

5. (12 pts) A baseball diamond is a square with side 27 m. A player runs from the first base to the second base at a rate of 5 m/s.

(a) (4 pts) At what rate is the player's distance from the third base changing when the player is 9m from the first base?

(b) (4 pts) At what rate is the angle θ changing at the moment in part (a)?

(c) (4 pts) The player slides into the second base at a rate of 4.5 m/s. At what rate is the angle θ changing as the player touches the second base?



Solution:

(a) Method 1:
Let x be the distance from the player to the second base, y be the distance from the player to the third base, then

$$y^2 = x^2 + 27^2 \qquad \text{(2 points)}$$

Differential both side by time t,

$$2y \cdot \frac{dy}{dt} = 2x \cdot \frac{dx}{dt} \implies \frac{dy}{dt} = (-)5 \cdot \frac{x}{y} \qquad \text{(2 points)}$$

Hence, when $x = 18, y = 9\sqrt{3}$,

$$\frac{dy}{dt} = (-)\frac{10}{\sqrt{13}}$$

Method 2:
Let y be the distance from the player to the third base, then for time t

$$y = \sqrt{27^2 + (18 - 5t)^2} \qquad \text{(2 points)}$$

Differential both side by t,

$$\frac{dy}{dt} = \frac{(-5)(18 - 5t)}{\sqrt{27^2 + (18 - 5t)^2}} \qquad \text{(2 points)}$$

Hence, when $t = 0$,

$$\frac{dy}{dt} = (-)\frac{10}{\sqrt{13}}$$

(b) Let x be the distance from second base to player,
$\tan\theta = \frac{x}{27}$ (1.5%)
By taking derivative with respect to t, we get
$(\sec\theta)^2\theta' = \frac{1}{27}x'$ (1.5%)

When $x(t) = 27 - 9 = 18$, $\sec\theta = \sqrt{13}/3$.
Since $x'(t) = -5$, when $x(t) = 18$, $\theta'(t) = (-5/27)(9/13) = -5/39$.(rad/s) (1%)

There is other solution:
$\sec(\theta(t)) = R(t)/27$, where $R(t)$ is the distance between the runner and the third base.
Similarly, taking derivative with respect to t, we get
$(\sec\theta)(\tan\theta)\theta' = R'/27$
When $R(t) = 9\sqrt{13}, R' = -10/\sqrt{13}, \sec\theta = \sqrt{13}/3, \tan\theta = 18/27$
Hence,$\theta' = -5/39$

(c) solution 1:
設跑者與二壘的距離為 t 的函數: $x(t)$。 則夾角的關係可以用 $\tan\theta(t)$ 表示。
$\tan\theta(t) = \dfrac{x(t)}{27}$ (1 point)
$\dfrac{d\tan(\theta(t))}{dt} = \sec^2\theta(t)\cdot\dfrac{d\theta}{dt} = \dfrac{1}{27}\cdot\dfrac{dx(t)}{dt}$ (1 point)
依題目所說向二壘移動的速率為 4.5 m/s,到達二壘時離二壘的距離為0。
故以
$\dfrac{dx(t)}{dt} = -4.5(m/s), x(t) = 0, \theta(t) = 0$ 代入 (1 point)
$\dfrac{d\theta(t)}{dt} = \dfrac{1}{27}\cdot(-4.5)\cdot 1 = -\dfrac{1}{6}$ (1 point)
solution 2:
設夾角 $\theta(t) = \tan^{-1}\dfrac{x}{27}$ (1 point)
$\theta'(t) = 1/(1 + (x/27)^2)\cdot(1/27)\cdot(dx(t))/t$ (1 point)
以 $x(t) = 0, (dx(t))/dt = -4.5$(m/s) 代入(1 point)
$\theta'(t) = 1\cdot(1/27)\cdot(-4.5) = -1/6$ (1 point)

[模02班] Suppose that it costs a manufacturer $C(x)$ dollars to produces x units of products, and $C(x)$ is differentiable for $x > 0$. When x units of products are produced, we call $C'(x)$ the "marginal cost" and $\frac{C(x)}{x}$ the "average cost".

(a) (4 pts) Write down the profit function, $P(x)$, when the manufacturer produces x units and sells them at a fixed price p_0 per unit. Show that when the profit obtains its maximum value at $x_1 > 0$, the marginal cost equals p_0.

(b) (4 pts) Assume that the average cost obtains its minimum value at $x_2 > 0$. Show that at $x = x_2$, the marginal cost equals the average cost.

(c) (4 pts) Suppose that the second derivative of the average cost is positive near x_2. Is the average cost greater than the marginal cost when x is near x_2 and $x < x_2$? What if $x > x_2$?

Solution:

(a) Let $P(x)$ be the profit, then

$$P(x) = px - C(x) \qquad \text{(2 points)}$$

Since $x_0 > 0$ and $x = x_0$ maximize $P(x)$, so

$$P'(x_0) = 0 \implies p - C'(x_0) = 0 \implies C'(x_0) = p \qquad \text{(2 points)}$$

Which means the marginal cost is p.

(b) Let $A(x) = \frac{C(x)}{x}$, then $A' = \frac{xC'(x) - C(x)}{x^2}$. (2%)

Because x_2 is in the interior of the domain of $A(x)$ and $A(x)$ is differentiable, Fermat's Theorem tells us that $A'(x_2) = 0$.

$\frac{x_2 C'(x_2) - C(x_2)}{(x_2)^2} = 0$

$C'(x_2) = \frac{C(x_2)}{x_2}$ (2%)

(c) $A(x) = (C(x))/x$, we know that $A''(x) > 0$,when near x_2.
By(b), we know that $A(x)$ is at its minimun at x_2. So we can derive:

$$A'(x) = \frac{xC'(x) - C(x)}{x^2} < 0 \text{ when } x < x_2.$$

$$A'(x) = \frac{xC'(x) - C(x)}{x^2} > 0 \text{ when } x > x_2.$$

Since $x^2 > 0$,

$$xC'(x) - C(x) < 0 \text{ when } x < x_2 \Rightarrow C'(x) < \frac{C(x)}{x}.(2point)$$

$$xC'(x) - C(x) > 0 \text{ when } x > x_2 \Rightarrow C'(x) > \frac{C(x)}{x}.$$

Hence, average cost > marginal cost, when $x < x_2$ average cost < marginal cost, when $x > x_2$. (2 point)

1 point for drawing graph correctly

6. (10 pts) Consider an isosceles triangle whose legs (the equal sides) have length ℓ and whose vertex angle is θ. As ℓ and θ vary, the area of the triangle stays the same. At which θ does ℓ attain its extreme value? Is this extreme value the maximum length or minimum length?

Solution:

(method 1) $A = \frac{1}{2} l^2 \sin\theta \quad 0 < \theta < \pi \quad$ (2 pts)

$\frac{dA}{d\theta} = 0 = \frac{1}{2} \times 2l \times l' \sin\theta + \frac{l^2}{2} \times \cos\theta$ (3 pts)

$\Rightarrow l' = \frac{-l}{2} \cot\theta$ (1 pt)

$l' = 0 \Leftrightarrow \theta = \frac{\pi}{2}$ for $0 < \theta < \pi$ (1 pt)

$l' < 0$ for $0 < \theta < \frac{\pi}{2}$

$l' > 0$ for $\frac{\pi}{2} < \theta < \pi$

by first derivative test for absolute extreme value test

l attain it's absolute minimum only when $\theta = \frac{\pi}{2}$

and l doesn't have absolute maximum in $0 < \theta < \pi$ (3 pts)

(method 2) $A = \frac{1}{2} l^2 \sin\theta \quad 0 < \theta < \pi \quad$ (2 pts)

since $\sin\theta \le 1$, and $\sin\theta = 1$ when $\theta = \frac{\pi}{2}$ in $0 < \theta < \pi$ (2 pts)

so l attain it's absolute minimum only when $\theta = \frac{\pi}{2}$ (2 pts)

since $\lim_{\theta \to 0^+} \sin\theta = \lim_{\theta \to \pi^-} \sin\theta = 0$

then $\lim_{\theta \to 0^+} l(\theta) = \lim_{\theta \to \pi^-} l(\theta) = \infty$ (2 pts)

so l doesn't have absolute maximum in $0 < \theta < \pi$ (2 pts)

7. (16 pts) Consider the function $f : (1, \infty) \to \mathbb{R}$, $f(x) = \ln x - \ln(\ln x)$.

(a) (4 pts) Compute $f'(x)$. Find the intervals in the domain of f on which f is increasing and those on which f is decreasing. What are the extreme values of f?

(b) (4 pts) Compute $f''(x)$. Find the intervals on which the graph of f is concave upward and those on which the graph of f is concave downward. Is there any inflection point of the curve $y = f(x)$?

(c) (5 pts) Find the vertical and horizontal asymptotes of the curve $y = f(x)$ if any. Find $\lim_{x\to\infty} \frac{f(x)}{x}$. Does the curve $y = f(x)$ have any slant asymptote?

(d) (3 pts) Draw the graph of $f(x)$.

Solution:

(a)

$$f'(x) = \frac{1}{x} - \frac{1}{x \ln x} = \frac{1}{x}\frac{\ln x - 1}{\ln x} \quad \text{(1 point)}$$

Note that $\ln x - 1 > 0$ if $x > e$, and $\ln x > 0$ if $x > 1$, hence

$$f'(x) = \begin{cases} > 0, & x > e \\ < 0, & 1 < x < e \\ = 0, & x = e \end{cases}$$

So the answer is f is increasing on $[e, \infty)$ and decreasing on $(1, e]$, and f has an extreme value 1 at $x = e$, which is a local(also global) minimum. (1 point for each. Answering strictly increasing/decreasing intervals: $(e, \infty)/(1, e)$ is also accepted.)

(b) From (a), we know that

$$f''(x) = \frac{d}{dx}\left[\frac{\ln x - 1}{x \ln x}\right] = \frac{\ln x - (\ln x - 1)(\ln x + 1)}{(x \ln x)^2} = -\frac{(\ln x)^2 - \ln x - 1}{(x \ln x)^2}. \quad \text{(1 pt)}$$

Let $t = \ln x$,

$$(\ln x)^2 - \ln x - 1 = t^2 - t - 1 = \left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right).$$

Hence, we can get

$$f''(x) > 0 \text{ when } \frac{1-\sqrt{5}}{2} < t < \frac{1+\sqrt{5}}{2} \text{ i.e. } \exp\left(\frac{1-\sqrt{5}}{2}\right) < x < \exp\left(\frac{1+\sqrt{5}}{2}\right).$$

It means that the graph of f is concave upward on $\left(1, \exp\left(\frac{1+\sqrt{5}}{2}\right)\right)$ since the domain of f is $(1, \infty)$. (1 pt)

Similarly,

$$f''(x) < 0 \text{ when } t < \frac{1-\sqrt{5}}{2} \text{ or } t > \frac{1+\sqrt{5}}{2} \text{ i.e. } x < \exp\left(\frac{1-\sqrt{5}}{2}\right) \text{ or } x > \exp\left(\frac{1+\sqrt{5}}{2}\right).$$

It means that the graph of f is concave downward on $\left(\exp\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$ since the domain of f is $(1, \infty)$. (1 pt)

According to the above, the inflection point of the curve $y = f(x)$ is

$$\left(\exp\left(\frac{1+\sqrt{5}}{2}\right), f\left(\exp\left(\frac{1+\sqrt{5}}{2}\right)\right)\right) \text{ or } \left(\exp\left(\frac{1+\sqrt{5}}{2}\right), \frac{1+\sqrt{5}}{2} - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right). \quad \text{(1 pt)}$$

Remark

1. Answering concave upward/concave downward intervals : $(1, \exp(\frac{1+\sqrt{5}}{2})]/[\exp(\frac{1+\sqrt{5}}{2}), \infty)$ is also accepted.
2. If you do not explain that f is concave upward/concave downward since $f''(x) > 0/f''(x) < 0$, you will lose 1 pt.
   Moreover, you will lose more points if you only explain that the inflection point of the curve $y = f(x)$ is $(\exp(\frac{1+\sqrt{5}}{2}), \frac{1+\sqrt{5}}{2} - \ln(\frac{1+\sqrt{5}}{2}))$ since $f''(\exp(\frac{1+\sqrt{5}}{2})) = 0$.
3. If you calculate the wrong derivative function, you will still get some points when you explain that the correct concepts.

(c) Notice that $\lim_{x\to a} f(x) = \lim_{x\to a}(\ln x - \ln(\ln x)) = \lim_{x\to a}(\ln \frac{x}{\ln x}) = \ln(\lim_{x\to a} \frac{x}{\ln x})$
(1 pt) $\lim_{x\to 1^+} f(x) = \ln(\lim_{x\to 1^+} \frac{x}{\ln x}) = -\ln 0^+ = \infty$
(1 pt) There is a vertical asymptote: $x = 1$

(1 pt) $\lim_{x\to\infty} f(x) = \ln(\lim_{x\to\infty} \frac{x}{\ln x}) \overset{L'H}{=} \ln(\lim_{x\to\infty} \frac{1}{\frac{1}{x}}) = \ln(\lim_{x\to\infty} x) = \infty$
(1 pt) There is NO horizontal asymtote.

(1 pt) $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln x}{x} - \lim_{x\to\infty} \frac{\ln(\ln x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{1}{x\ln x} = 0 - 0 = 0$
(1 pt) Since $m = \lim_{x\to\infty} \frac{f(x)}{x} = 0$, $y = f(x)$ have NO slant asymptote.
Minus up to 5 points.

(d) (1 pt) Label points: $(e, 1)$ as minimum, $x = e^{\frac{1+\sqrt{5}}{2}}$ as inflection point.
(Note: One SHOULD clearly label $(e, 1)$. Inflection point only $x - axis$ is needed.
Get full credits only if two points are labeled.)
(1 pt) $x = 1$ as vertical asymptote.
(1 pt) Increasing/decreasing interval, Concavity interval
(Note: Get full credits only if all interval are right.)