1071微甲及模組班期末考解答和評分標準

1. Evaluate the integrals:

(a) (6%) $\int \tan^{\frac{1}{3}} x \sec^4 x \, dx$ (b) (9%) $\int \frac{4}{x(x^2+2x+2)}\, dx$

Solution:

(a)

Let $u = \tan^{\frac{1}{3}} x$ (到此步驟得 1 分)

$\Rightarrow du = \frac{1}{3}\tan^{-\frac{2}{3}} x \sec^2 x dx$ (到此步驟得 2 分)

$\Rightarrow \sec^2 x dx = 3u^2 du$

Note $u^6 = \tan^2 x = \sec^2 x - 1$

$\Rightarrow \sec^2 x = 1 + u^6$ (到此步驟得 3 分)

Hence

$I = \int \left(\tan^{\frac{1}{3}} x\right)\left(\sec^4 x\right) dx$

$= \int u \cdot (1+u^6) \cdot 3u^2 du$ (到此步驟得 4 分)

$= \int 3(u^3 + u^9) du$

$= 3\left(\frac{u^4}{4} + \frac{u^{10}}{10}\right) + C$ (到此步驟得 5 分)

$\Rightarrow I = \frac{3}{4}\tan^{\frac{4}{3}} x + \frac{3}{10}\tan^{\frac{10}{3}} x + C$ (到此步驟得 6 分)

(b)

Write $\frac{4}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$ (到此步驟得 1 分)

$\Rightarrow 4 = A(x^2+2x+2) + (Bx+C)x$

$\Rightarrow 4 = 2A + 0 \Rightarrow A = 2$ (到此步驟得 1 分)

$0 = 2A + C \Rightarrow C = -4$ (到此步驟得 2 分)

$0 = A + B \Rightarrow B = -2$ (到此步驟得 3 分)

Perform

$I = \int \frac{4}{x(x^2+2x+2)} dx = \int \frac{2}{x} dx + \int \frac{-2x-4}{x^2+2x+2} dx = I_1 + I_2$ (到此步驟得 4 分)

Hence $I_1 = 2\ln|x|$ (到此步驟得 4 分)

$I_2 = \int \frac{-2(x+1)-2}{(x+1)^2+1} dx$ (到此步驟得 5 分)

$= \int \frac{-2(x+1)}{(x+1)^2+1} dx + \int \frac{-2}{(x+1)^2+1} dx$ (到此步驟一半得 5 分，全對得 6 分)

$= -\ln|(x+1)^2+1| - 2\tan^{-1}(x+1)$ (到此步驟一半得 7 分，全對得 8 分)

In Summary

$I = I_1 + I_2 = 2\ln|x| - \ln|(x+1)^2+1| - 2\tan^{-1}(x+1) + C$ (到此步驟得 9 分)

2. (12%) Let R be the region bounded by the curves $y = e^{x^2}$, $y = 0$, $x = \sqrt{a}$, and $x = \sqrt{a+1}$, where $a > 0$. Find the number a such that the area of R attains the minimum value.

Solution:

$$A = \int_{\sqrt{a}}^{\sqrt{a+1}} e^{x^2}\,dx \qquad \text{(寫出這一式得 3 分)}$$

$$\frac{dA}{da} = \frac{d\int_{\sqrt{a}}^{\sqrt{a+1}} e^{x^2}\,dx}{da} = 0 \qquad \text{(到此步驟得 6 分)}$$

$$= \frac{e^{a+1}}{2\sqrt{a+1}} - \frac{e^{a}}{2\sqrt{a}} \qquad \text{(到此步驟得 8 分)}$$

$$\Rightarrow a = \frac{1}{e^2-1} \qquad \text{(到此步驟得 10 分)}$$

$$\text{check } \frac{d\left(\frac{e^{a+1}}{2\sqrt{a+1}} - \frac{e^{a}}{2\sqrt{a}}\right)}{da} > 0, \text{ when } a = \frac{1}{e^2-1} \qquad \text{(到此步驟得 12 分)}$$

3. (a) (9%) Determine whether $\int_1^\infty \frac{\tan^{-1} x}{x^2} dx$ converges or diverges. Evaluate the value if it converges.

(b) (6%) Determine whether $\int_{-1}^1 \frac{\tan^{-1} x}{x^2} dx$ converges or diverges. Evaluate the value if it converges.

Solution:

(a) (Method 1)

$\int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2} dx$ (Definition of improper integral: 1 point)

denote $u = \tan^{-1} x$ $\tan u = x \Rightarrow dx = \sec^2 u du$

$\therefore \int_1^b \frac{\tan^{-1} x}{x^2} dx = \int_{\frac{\pi}{4}}^{\tan^{-1} b} \frac{u \sec^2 u}{\tan^2 u} du$ (change variables and upper lower limits: 2 points)

$= \int_{\frac{\pi}{4}}^{\tan^{-1} b} u \csc^2 u du$

$= -u \cot u\Big|_{\frac{\pi}{4}}^{\tan^{-1} b} + \int_{\frac{\pi}{4}}^{\tan^{-1} b} \cot u du$ (integration by part: 2 points)

$= -u \cot u + \ln|\sin u|\Big|_{\frac{\pi}{4}}^{\tan^{-1} b}$

$= -\frac{\tan^{-1} b}{b} + \ln \frac{b}{\sqrt{1+b^2}} + \frac{\pi}{4} - \ln \frac{1}{\sqrt{2}}$ (2 points)

$\lim_{b\to\infty} \frac{\tan^{-1} b}{b} = 0$ because $\lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$

$\lim_{b\to\infty} \ln \frac{b}{\sqrt{1+b^2}} = \ln 1 = 0$

(discuss limits and converge + correct integral: 2 points)

$\therefore \int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2} dx = \frac{\pi}{4} + \frac{\ln 2}{2}$ converges.

Remark 1. *If do not follow the definition of improper integral, but write the final step as* $-\frac{1}{x}\tan^{-1} x + \ln \frac{x}{\sqrt{1+x^2}}\Big|_1^\infty = \frac{\pi}{4} - \ln\frac{1}{\sqrt{2}}$ *without discussing how the limit is obtained, it will be deducted 2 points even if the answer is correct.*

Remark 2. *If use comparison theorem to conclude that the integral converges without any mistake, it will get 3 points. (The first 1 point + last 2 points)*

(Method 2)

$$\int \frac{\tan^{-1} x}{x^2} dx = -\frac{1}{x}\tan^{-1} x + \int \frac{1}{x(x^2+1)} dx \qquad \text{(2 points)}$$

$$= -\frac{1}{x}\tan^{-1} x + \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \qquad \text{(2 points)}$$

$$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

$$= -\frac{1}{x}\tan^{-1} x + \ln\frac{|x|}{\sqrt{x^2+1}} + C \qquad \text{(2 points)}$$

$$\therefore \int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2} dx \qquad \text{(1 point)}$$

$$= \lim_{b\to\infty} -\frac{1}{x}\tan^{-1} x + \ln\frac{x}{\sqrt{x^2+1}}\Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{\tan^{-1} b}{b} + \ln\frac{b}{\sqrt{b^2+1}} + \tan^{-1} 1 - \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \ln\frac{1}{\sqrt{2}}$$

$$\left(\because \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2} \ \therefore \lim_{b\to\infty} \frac{\tan^{-1} b}{b} = 0 \text{ and } \lim_{b\to\infty} \ln\frac{b}{\sqrt{b^2+1}} = \ln 1 = 0\right) \qquad \text{(2 points)}$$

(b) $\dfrac{\tan^{-1} x}{x^2}$ is improper at $x = 0$

$\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x}{x^2} dx$

$\displaystyle = \int_{-1}^{0} \frac{\tan^{-1} x}{x^2} dx + \int_{0}^{1} \frac{\tan^{-1} x}{x^2} dx$ (1 point)

$\displaystyle = \lim_{a \to 0^-} \int_{-1}^{a} \frac{\tan^{-1} x}{x^2} dx + \lim_{b \to 0^+} \int_{b}^{1} \frac{\tan^{-1} x}{x^2} dx$ (1 point)

$$\int_{b}^{1} \frac{\tan^{-1} x}{x^2} dx = -u \cot u + \ln|\sin u|\Big|_{\tan^{-1} u}^{\frac{\pi}{4}} \quad \text{from (a)}$$

$$= -\frac{\pi}{4} + \ln \frac{1}{\sqrt{2}} + \frac{\tan^{-1} b}{b} - \ln \frac{b}{\sqrt{1+b^2}}$$

(1 point) (also can look at $\displaystyle\int_{-1}^{a} \frac{\tan^{-1} x}{x^2} dx$)

$\displaystyle\lim_{b \to 0} \frac{\tan^{-1} b}{b} \underset{\frac{0}{0}}{=} \lim_{b \to 0} \frac{\frac{1}{1+b^2}}{1} = 1$ (1 point)

$\displaystyle\lim_{b \to 0} \ln \frac{b}{\sqrt{1+b^2}} = -\infty$ (1 point)

Because $\displaystyle\lim_{b \to 0^+} \int_{b}^{1} \frac{\tan^{-1} x}{x^2} dx = \infty$

$\displaystyle\therefore \int_{-1}^{1} \frac{\tan^{-1} x}{x^2} dx$ diverges (1 point)

4. (a) (8%) Show that $y = 0$ is an orthogonal trajectory of the family of curves $x^2 + \frac{y^2}{k} = 1$, where $k > 0$ is an arbitrary constant. Find the orthogonal trajectories of the same family of curves when $y \neq 0$.

(b) (8%) Find $u(t)$ that satisfies the ordinary differential equation

$$u'(t) + \ln(t)u(t) = e^{-t\ln(t)}, \quad t > 0$$

and the condition

$$\lim_{t\to 0^+} u(t) = 2.$$

Solution:

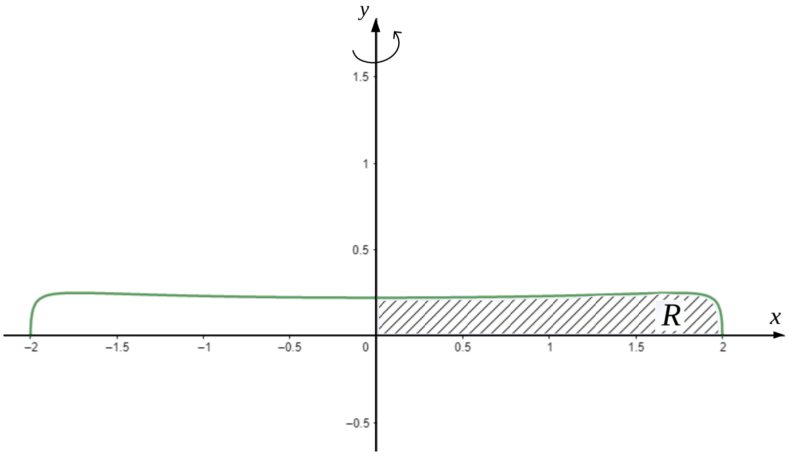
(a)

- For whatever $k > 0$, the associated curve passes through the points $(\pm 1, 0)$, and $x^2 + \frac{y^2}{k} = 1 \Rightarrow 2xdx + \frac{2y}{k}dy = 0 \Rightarrow dx = 0$ at these points $\Rightarrow$ vertical tangents thereon (1%)
- The line $y = 0$ also passes through the two points above, and of course perpendicular to the local vertical tangent $\Rightarrow y = 0$ is an orthogonal trajectory (1%)
- $2xdx + \frac{2y}{k}dy = 0 \Rightarrow (dx, dy)$ parallel to $(x, \frac{y}{k}) = (x, \frac{1-x^2}{y})$ at any (x, y) with $y \neq 0$ on the orthogonal family (2%)
- $\Rightarrow \frac{dx}{x} = \frac{ydy}{1-x^2} \Rightarrow \int \frac{1-x^2}{x}\,dx = \int y\,dy$ (2%)
- $\Rightarrow \ln|x| - \frac{x^2}{2} + \frac{C}{2} = \frac{y^2}{2}$ or $x^2 + y^2 - \ln x^2 = C$ (1%)
- $y \neq 0 \Rightarrow x^2 < 1 \Rightarrow x^2 - \ln x^2 > 1 \Rightarrow C > 1$ (1%)

(b)

- $I = e^{\int^t \ln s\,ds} = e^{t\ln t - \int^t \frac{s}{s}\,ds}$ (1%)
- $\Rightarrow I = e^{t(\ln t - 1)}$ (1%)
- $\Rightarrow u(t) = \frac{C + \int^t I \cdot e^{-s\ln s}\,ds}{I}$ (1%)
- $\Rightarrow u(t) = e^{t(1-\ln t)}(C - e^{-t})$ (2%)
- $\lim_{t\to 0^+} u(t) = 2 \Rightarrow 2 = C - 1$ (2%)
- $\Rightarrow u(t) = e^{t(1-\ln t)}(3 - e^{-t})$ (1%)

5. (a) (8%) Let R be the region bounded by the curves $y = \dfrac{\sqrt{4-x^2}}{(1+\sqrt{4-x^2})^2}$, $y = 0$, $x = 0$, and $x = 2$. Find the volume of the solid obtained by rotating R about the y-axis.



Solution:

Using the *method of cylindrical shells*, the volume of the solid in question is given by

$$V = 2\pi \int_0^2 x \frac{\sqrt{4-x^2}}{\left(1+\sqrt{4-x^2}\right)^2}\, dx\ .$$

(correct formula: 2 points)

Let $u := \sqrt{4-x^2}$, which gives $u^2 = 4 - x^2$, thus yielding $u\,du = -x\,dx$. We then obtain from the *substitution rule* that

$$V = -2\pi \int_2^0 \frac{u^2\, du}{(1+u)^2} = 2\pi \int_0^2 \frac{u^2\, du}{(1+u)^2}\ .$$

Finally,

integration by parts gives

$$\begin{aligned} V &= 2\pi \int_0^2 u^2\, d\left(\frac{-1}{1+u}\right) \\ &= 2\pi \left[\frac{-u^2}{1+u}\right]_0^2 + 2\pi \int_0^2 \frac{2u}{1+u}\, du \\ &= -\frac{8\pi}{3} + 4\pi \int_0^2 \left(1 - \frac{1}{1+u}\right) du \\ &= -\frac{8\pi}{3} + 4\pi \left[\, u - \ln(1+u)\,\right]_0^2 \\ &= \frac{16\pi}{3} - 4\pi \ln 3\ . \end{aligned}$$

partial fraction decomposition gives

$$\begin{aligned} V &= 2\pi \int_0^2 \frac{((1+u)-1)^2}{(1+u)^2}\, du \\ &= 2\pi \int_0^2 \left(1 - \frac{2}{1+u} + \frac{1}{(1+u)^2}\right) du \\ &= 2\pi \left[\, u - 2\ln(1+u) - \frac{1}{1+u}\,\right]_0^2 \\ &= \frac{16\pi}{3} - 4\pi \ln 3\ . \end{aligned}$$

(computation + answer: 5+1 points)

(b) (8%) Find the area of the infinite surface generated by rotating the curve $y = e^{-x}$, $0 \le x < \infty$, about the x-axis if it is finite (show explicitly in your calculation that you are using the definition of improper integrals).

Solution:

The area of the surface in question is given by

$$\begin{aligned} S = 2\pi \int_0^\infty e^{-x}\sqrt{1+(-e^{-x})^2}\, dx &= 2\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}\, dx \\ &= 2\pi \lim_{t\to\infty} \int_0^t e^{-x}\sqrt{1+e^{-2x}}\, dx\ . \end{aligned}$$

(correct formula: 2 points)

(showing the knowledge of improper integrals: 1 point)

Let $u := e^{-x}$, which yields $du = -e^{-x}\,dx$. The *substitution rule* then gives

$$S = -2\pi \lim_{t\to\infty} \int_1^{e^{-t}} \sqrt{1+u^2}\,du = 2\pi \lim_{t\to\infty} \int_{e^{-t}}^1 \sqrt{1+u^2}\,du$$
$$= 2\pi \int_0^1 \sqrt{1+u^2}\,du \ ,$$

where the last integral is a *proper integral* (which follows from the facts that $\lim_{t\to\infty} e^{-t} = 0$ and that the function $\int_x^1 \sqrt{1+u^2}\,du$ in x is continuous on $[0,1]$ by the *Fundamental Theorem of Calculus*, as the integrand $\sqrt{1+u^2}$ is continuous on the interval $[0,1]$).

As a result, the surface area in question is given by

$$\begin{aligned}
S &= 2\pi \int_0^1 \sqrt{1+u^2}\,du \\
&= 2\pi \left[\, u\sqrt{1+u^2}\,\right]_0^1 - 2\pi \int_0^1 \frac{u^2}{\sqrt{1+u^2}}\,du \\
&= 2\sqrt{2}\,\pi - 2\pi \int_0^1 \sqrt{1+u^2}\,du + 2\pi \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du \\
&= \sqrt{2}\,\pi + \pi \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du \\
&\overset{(u:=\tan\theta)}{=} \sqrt{2}\,\pi + \pi \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta = \sqrt{2}\,\pi + \pi \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta \\
&= \sqrt{2}\,\pi + \pi \left[\, \ln|\sec\theta + \tan\theta|\,\right]_0^{\frac{\pi}{4}} \\
&= \sqrt{2}\,\pi + \pi \ln\!\left(\sqrt{2}+1\right) .
\end{aligned}$$

(computation + answer: 4+1 points)

[經濟系]

(b) (8%) Let $\tilde{R}$ be the region under the curve $y = \dfrac{1}{(1+x^2)^{\frac{3}{4}}}$, above the line $y = 0$, and to the right of $x = 1$. Find the volume of the solid obtained by rotating $\tilde{R}$ about the x-axis if it is finite (show explicitly in your calculation that you are using the definition of improper integrals).

Solution:

$$V = \int_1^\infty \pi (y(x))^2 dx$$

$$= \pi \int_1^\infty \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$$ (2 points. Formula for the volume)

$$\int_1^\infty \frac{1}{(1+x^2)^{\frac{3}{2}}} dx = \lim_{t\to\infty} \int_1^t \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$$ (1 point. Use the definition of improper integrals.)

$$= \lim_{t\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} t} \frac{1}{\sec^3\theta} \sec^2\theta d\theta$$ (3 points:)

Let $x = \tan\theta$ (Trigonometric substitution $x = \tan\theta \Rightarrow$ 1 point.)

$-\dfrac{\pi}{2} < \theta < \dfrac{\theta}{2}$ (Correct integrand $\dfrac{1}{\sec^3\theta}\sec^2\theta \Rightarrow$ 1 point.)

$dx = \sec^2\theta d\theta$ (Correct upper and lower bounds $\Rightarrow$ 1 point)

$$= \lim_{t\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} t} \cos\theta d\theta$$

$$= \lim_{t\to\infty} \left(\sin(\tan^{-1} t) - \sin(\frac{\pi}{4})\right)$$ (1 point. Trigonometric integration.)

$$= 1 - \frac{1}{\sqrt{2}}, \quad V = \pi(1 - \frac{1}{\sqrt{2}})$$ (1 point. For taking limits and the final answer.)

[電機系]

Consider the differential equation $\begin{cases} y''(t) + 2y'(t) + y(t) = f(t) \\ y(0) = 1, y'(0) = 0 \end{cases}$, where $f(t) = \begin{cases} 2\cos t, & \text{for } 0 \le t < \pi \\ 0, & \text{for } t \ge \pi \end{cases}$.

(a) (2%) Find the general solution of the related homogeneous equation $y''(t) + 2y'(t) + y(t) = 0$.

(b) (4%) Write $f(t)$ in terms of the unit step function $\mathcal{U}(t-\pi) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & t \ge \pi \end{cases}$. Compute $\mathcal{L}\{f(t)\}$, the Laplace transform of $f(t)$.

(c) (4%) Let $Y(s)$ be the Laplace transform of the solution $y(t)$. Apply Laplace transform to the differential equation and solve for $Y(s)$.

(d) (6%) Solve the differential equation.

Solution:

(a) The characteristic equation of $y''(t) + 2y'(t) + y(t) = 0$ is

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1 \text{ (repeated root)} \qquad \text{(1 point)}$$

Hence the general solution of the homogeneous equation is

$$c_1 e^{-t} + c_2 t e^{-t}, \text{ where } c_1, c_2 \text{ are constants} \qquad \text{(1 point)}$$

(b)

$$f(t) = 2\cos t(1 - \mathcal{U}(t-\pi)) \qquad \text{(1 point)}$$

Solution 1:

$$\begin{aligned} &\mathcal{L}\{f(t)\} = 2\mathcal{L}\{\cos t\} - 2\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\} \\ &= \frac{2s}{s^2+1} - 2e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\} = \frac{2s}{s^2+1} - 2e^{-\pi s}\mathcal{L}\{-\cos t\} \\ &= 2\frac{s}{s^2+1}[1 + e^{-\pi s}] \end{aligned}$$

$$\boxed{\text{Partial Credits:}}\ \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \qquad \text{(1 point)}$$

$$\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\} = -e^{-\pi s}\frac{s}{1+s^2} \qquad \text{(2 points)}$$

Solution 2:

$$\mathcal{L}\{f(t)\} = \int_0^{\pi} e^{-st} \cdot 2\cos t dt \qquad \text{(1 point. For using the definition of Laplace transform)}$$

$$\begin{aligned} &= 2\left[\sin t e^{-st}\Big|_{t=0}^{t=\pi} + \int_0^{\pi} s \cdot \sin t \cdot e^{-st} dt\right] \\ &= 2s\left[-\cos t e^{-st}\Big|_{t=0}^{t=\pi} - \int_0^{\pi} (-\cos t)(-s)e^{-st} dt\right] \\ &= 2s\left(e^{-\pi s} + 1 - \frac{s}{2}\mathcal{L}\{f(t)\}\right) \\ &\Rightarrow \mathcal{L}\{f(t)\} = \frac{2s}{1+s^2}(1 + e^{-\pi s}) \end{aligned} \qquad \text{(2 points)}$$

(c) Apply Laplace transform to the differential equation

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\} \qquad \text{(1 point. Linearity of Laplace transform.)}$$

$$\Rightarrow (s^2 Y(s) - sy(0) - y'(0)) + 2(sY(s) - y(0)) + Y(s) = \frac{2s}{1+s^2}(1 + e^{-\pi s})$$

$$\boxed{\text{Partial credits:}}\ \mathcal{L}\{y'\} = sY(s) - y(0) \qquad \text{(1 point)}$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) \qquad \text{(1 point)}$$

$$(s^2 + 2s + 1)Y(s) = s + 2 + \frac{2s}{s^2+1}(1 + e^{-\pi s})$$

$$Y(s) = \frac{s+2}{(s+1)^2} + \frac{2s}{(s+1)^2(s^2+1)}(1 + e^{-\pi s}) \qquad \text{(1 point)}$$

(d) Decompose $\dfrac{s+2}{(s+1)^2}$ and $\dfrac{2s}{(s+1)^2(s^2+1)}$ as sum of partial fractions.

$$\frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \qquad \text{(1 point)}$$

$$\frac{2s}{(s+1)^2(s^2+1)} = \frac{1}{s^2+1} - \frac{1}{(s+1)^2} \qquad \text{(1 point)}$$

$$Y(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \left(\frac{1}{s^2+1} - \frac{1}{(s+1)^2}\right)(1+e^{-\pi s})$$

$$= \frac{1}{s+1} + \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{(s+1)^2}\right)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} + \mathcal{L}^{-1}\{e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{(s+1)^2}\right)\}$$

$$= e^{-t} + \sin t + \mathcal{U}(t-\pi)\left[\sin(t-\pi) - (t-\pi)e^{-(t-\pi)}\right] \qquad \text{(← Good as the final answer.)}$$

$$= e^{-t} + \sin t + \mathcal{U}(t-\pi)\left(-\sin t - e^{-\pi}(t-\pi)e^{-t}\right) \qquad \text{(← Good as the final answer.)}$$

$$= \begin{cases} e^{-t} + \sin t, & \text{for } 0 \le t < \pi \\ (1+\pi e^{\pi})e^{-t} - e^{\pi} t e^{-t}, & \text{for } t \ge \pi \end{cases} \qquad \text{(← Good as the final answer.)}$$

Partial credits:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad \text{(1 point)}$$

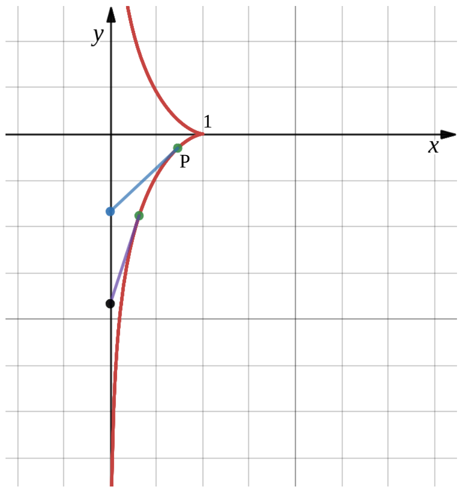
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \qquad \text{(1 point)}$$

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\} = \mathcal{U}(t-\pi)\sin(t-\pi) \qquad \text{(1 point)}$$

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s+1)^2}\right\} = \mathcal{U}(t-\pi)(t-\pi)e^{-(t-\pi)} \qquad \text{(1 point)}$$

6. A tractrix (the curve below) describes the path an obstinate dog takes when its master walks along the y-axis. One way to parameterize the curve is by means of

$$(x(t),\ y(t)) = (\sin t,\ \cos t + \ln(\csc t - \cot t)), \quad 0 < t < \pi.$$



(a) (8%) Find the slope of the tangent line at a point $P = (x(t_0), y(t_0))$ on the curve, and show that the distance between P and the y-intercept of the tangent line at P is independent of $t_0 \in (0, \pi)$.

(b) (6%) Compute the arc length of the part of the tractrix from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{2}$.

Solution:

(a) (8 points in total): $dy/dx = \dfrac{dy/dt}{dx/dt}$ (two points)

Now compute dy/dt and dx/dt

$dy/dt = -\sin(t) + \dfrac{1-\cos(t)}{\sin(t) - \sin(t)\cos(t)} = -\sin(t) + \csc(t)$ (one point)

$dx/dt = \cos(t)$ so that

$dy/dx = -\tan(t) + \dfrac{\csc(t)}{\cos(t)}$

Finally the slops is $-\tan(t_0) + \dfrac{\csc(t_0)}{\cos(t_0)}$ (one point)

The equation of the tangent line at t is $y - y(t) = m(x - x(t))$

so the y–intercept is $-mx(t) + y(t)$ (one point)

Thus if we denote the distance by D, we have $D^2 = (x(t))^2 + (y(t) - (-mx(t) + y(t)))^2$

$= x(t)^2(1 + m^2)$, where $m = -\tan(t) + \dfrac{\csc(t)}{\cos(t)}$ (one point)

Direct computation shows $D^2 = \sin(t)^2(\dfrac{\csc(t)^2 - 1}{\cos(t)^2}) = 1$ (two points)

Thus D is a constant, independent of t.

(b) (6 points in total): $L = \displaystyle\int_{\pi/4}^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$ (two points)

$= \displaystyle\int_{\pi/4}^{\pi/2} \sqrt{\cos(t)^2 + \sin(t)^2 - 2 + \csc(t)^2}dt$

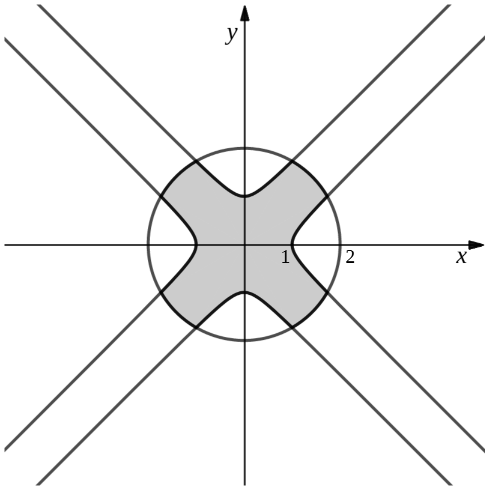
$= \displaystyle\int_{\pi/4}^{\pi/2} \cot(t)dt$ (two points)

$= \ln|\sin(t)||_{\pi/4}^{\pi/2} = \ln(\sqrt{2})$ (two points)

7. (a) (3%) Find $f(\theta)$ such that $r^2 = f(\theta)$ is the polar equation of the curve given by $x^2+y^2=(x^2-y^2)^2$, $(x,y)\neq(0,0)$.

(b) (3%) Find the points of intersection of the curves $x^2+y^2=(x^2-y^2)^2$ and $r=2$ (express the intersection points in terms of polar coordinates).

(c) (6%) Find the area of the region containing the origin and bounded by the curves $x^2+y^2=(x^2-y^2)^2$ and $r=2$ (the shaded region in the figure below).



Solution:

(a)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \text{(1 point)}$$

$x^2+y^2=(x^2-y^2)^2 \Rightarrow r^2 = r^4(\cos^2\theta-\sin^2\theta)^2 = r^4(\cos 2\theta)^2$

For $(x,y)\neq(0,0)$, $r\neq 0$.

Hence $r^2 = r^4(\cos 2\theta)^2 \Rightarrow 1 = r^2(\cos 2\theta)^2 \Rightarrow r^2 = (\sec 2\theta)^2 \begin{cases} \text{or } r^2 = \dfrac{2}{1+\cos 4\theta} \\ \text{or } r^2\cos^2 2\theta = 1 \\ \text{or } r^2(\cos^2\theta-\sin^2\theta)^2 = 1 \end{cases}$ (2 points)

(b) To find the points of intersection of the curve and $r=2$, we need to solve the equations

$$\begin{cases} r = \sec 2\theta \\ r = 2 \end{cases} \text{ or } \begin{cases} r = -\sec 2\theta \\ r = 2 \end{cases} \qquad \text{(1 points)}$$

Hence $2\theta = \frac{\pi}{3}, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi, \frac{7}{3}\pi, \frac{8}{3}\pi, \frac{10}{3}\pi$ or $\frac{11}{3}\pi$

$\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2}{3}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi$ or $\frac{11}{6}\pi$ (1 point)

$\xrightarrow{\text{(1 point)}}$ The points of intersection are $(2,\frac{\pi}{6})$ $(2,\frac{\pi}{3})$ $(2,\frac{2}{3}\pi)$ $(2,\frac{5}{6}\pi)$ $(2,\frac{7}{6}\pi)$ $(2,\frac{4}{3}\pi)$ $(2,\frac{5}{3}\pi)$ $(2,\frac{11}{6}\pi)$ (1 point)

(用直角坐標且正確: 扣 1 分，(r,θ) 座標寫反不扣分)

Notation:
(1) 只列出一個聯立方程式，解出一半交點: 2 points
(2) 只列出一個聯立方程式，解出一個交點或少於一半交點: 1 point
(3) 無過程，但列出正確的交點: 扣 1 point

(c) Solution 1:
By the symmetry, we only need to compute the area of the shaded region in the first quadrant and multiply

it by 4.

$$A = 4\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(\sec^2 2\theta)d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cdot 2^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(\sec^2 2\theta)d\theta\right] \qquad \text{(4 points)}$$

$$= 4\left[\frac{1}{4}\tan 2\theta\Big|_{\theta=0}^{\frac{\pi}{6}} + 2\times\frac{\pi}{6} + \frac{1}{4}\tan 2\theta\Big|_{\theta=\frac{\pi}{3}}^{\frac{\pi}{2}}\right] \qquad \text{(1 point)}$$

$$= \left(\tan\frac{\pi}{3} + \frac{4}{3}\pi + \tan\pi - \tan(\frac{2}{3}\pi)\right)$$

$$= \frac{4}{3}\pi + 2\sqrt{3} \qquad \text{(1 point)}$$

Partial credits:

$$A = 4\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(\sec^2 2\theta)d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cdot 2^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(\sec^2 2\theta)d\theta\right]$$

4 points : 1 point : For the formula $A = \frac{1}{2}\int (f(\theta))^2 d\theta$

1 point: For correctly divide the interval $[0, \frac{\pi}{2}]$ into $[0, \frac{\pi}{6}] \cup [\frac{\pi}{6}, \frac{\pi}{3}] \cup [\frac{\pi}{3}, \frac{\pi}{2}]$.

2 points: For correct integrands in each intervals.

Solution 2:

First compute the area of the white region inside the circle $r = 2$.

By symmetry, the area of the white region $= 8\int_0^{\frac{\pi}{6}} \frac{1}{2}(2^2 - \sec^2 2\theta)d\theta$ (3 points)

$$= 4\left[4\times\frac{\pi}{6} - \frac{1}{2}\tan 2\theta\Big|_0^{\frac{\pi}{6}}\right]$$

(1 point, for triagonometric integration)

$$= \frac{8}{3}\pi - 2\sqrt{3} \qquad \text{(1 point)}$$

Partial credits: $8\int_0^{\frac{\pi}{6}} \frac{1}{2}(2^2 - \sec^2 2\theta)d\theta$ (3 points):
1 point: For the formula $A = \frac{1}{2}\int (f(\theta))^2 d\theta$
1 point: For correct upper bound and lower bound
1 point: For correct integrand.

Hence the area of the shaded region is $\pi\cdot(2)^2 - \left(\frac{8}{3}\pi - 2\sqrt{3}\right) = \frac{4}{3}\pi + 2\sqrt{3}$ (1 point)