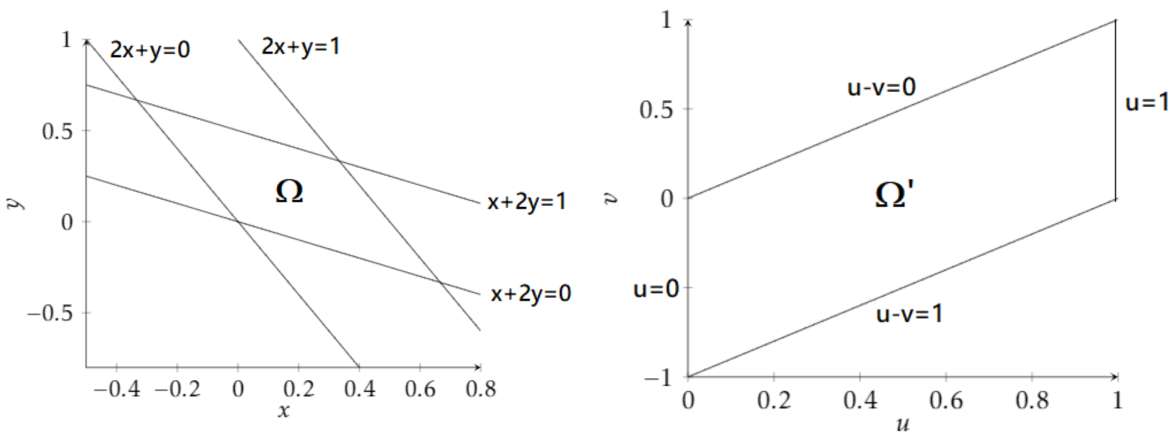
1062微乙01-05班期中考解答和評分標準

1. (13%) 求 $\iint_\Omega (x-y)^{20}\,dA$，其中 Ω 是由 $2x+y=0$, $2x+y=1$, $x+2y=0$ 及 $x+2y=1$ 所圍成的平行四邊形。

Solution:



(2pt)

Let $x+2y=u$, $x-y=v$ that is $x=\frac{1}{3}(u+2v)$, $y=\frac{1}{3}(u-v)$ ...(2 pt)

$$|\ J\ | = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix}\right| = \frac{1}{3} ...(2\ pt)$$

Note that the new region Ω' will enclosed by:

$2x+y=0 \to u-v=0$

$2x+y=1 \to u-v=1$

$x+2y=0 \to u=0$

$x+2y=1 \to u=1$

So we have

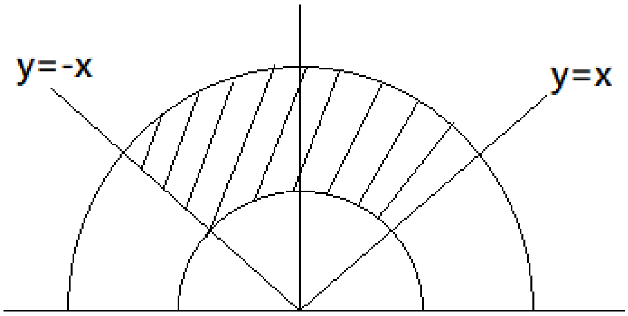
$$\Omega' = \begin{cases} u-1 \le v \le u \\ 0 \le u \le 1\ ...(1\ pt) \end{cases}$$

$$\begin{aligned} \iint_\Omega (x-y)^{20} dA &= \iint_{\Omega'} v^{20}\frac{1}{3} dA = \frac{1}{3}\int_0^1 \int_{u-1}^u v^{20} dv du = ...(1\ pt) \\ &= \frac{1}{3}\int_0^1 [\frac{1}{21}v^{21}]_{v=u-1}^{v=u} du = \frac{1}{63}\int_0^1 u^{21} - (u-1)^{21} du ...(2\ pt) \\ &= \frac{1}{63}\{[\frac{u^{22}}{22}]_{u=0}^{u=1} - [\frac{(u-1)^{22}}{22}]_{u=0}^{u=1}\} ...(2pt) \\ &= \frac{1}{63}(\frac{1}{22}+\frac{1}{22}) = \frac{1}{693} ...(1\ pt) \end{aligned}$$

2. (12%) 計算二重積分

$$\iint_{\Omega} e^{-(x^2+y^2)}\, dA, \qquad \Omega = \{(x,y) : 1 \le x^2+y^2 \le 2,\, y \ge 0,\, y \ge x \ge -y\}.$$

Solution:



...(2pt)

Let $x = r\cos\theta,\ y = r\sin\theta$...(2 pt)

$$|\ J\ | = |\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}| = |\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}| = r ...(2\ pt)$$

$y = x \to \theta = \frac{\pi}{4}$

$y = -x \to \theta = \frac{3\pi}{4}$

So we have:

$$\Omega = \begin{cases} 1 \le r \le \sqrt{2} \\ \frac{\pi}{4} \le u \le \frac{3\pi}{4} \end{cases} ...(1\ pt)$$

$$\iint_{\Omega} e^{-(x^2+y^2)} dA = \iint_{\Omega} e^{-r^2} r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{1}^{\sqrt{2}} e^{-r^2} r dr d\theta ...(1\ pt)$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} [\frac{-1}{2} e^{-r^2}]_{r=1}^{r=\sqrt{2}} d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} e^{-1} - \frac{1}{2} e^{-2} d\theta ...(2\ pt)$$

$$= \frac{1}{2}(e^{-1} - e^{-2})\frac{\pi}{2} = \frac{\pi}{4}(e^{-1} - e^{-2}) ...(2\ pt)$$

3. (13%) 求曲線 $10x^2 + 12xy + 5y^2 = 14$ 上和原點 $(0,0)$ 相距之最近點及最遠點。

Solution:

assume $f(x) = \sqrt{x^2+y^2}$

using Lagrange equation

$$\begin{cases} 2x = \lambda(20x+12y) \\ 2y = \lambda(12x+10y) \\ 10x^2+12xy+5y^2 = 14 \end{cases} \Rightarrow \frac{x}{y} = \frac{10x+6y}{6x+5y} \Rightarrow 6x^2-5xy-6y^2 = 0 \tag{1}$$

$\Rightarrow (3x+2y)(2x-3y) = 0$

condtition 1: $3x+2y=0$

$\frac{40}{9}y^2 - 8y^2 + 5y^2 = 14$

$y = \pm 3\sqrt{\frac{14}{13}}$

$x = \mp 2\sqrt{\frac{14}{13}}$

$f(\mp 2\sqrt{\frac{14}{13}}, \pm 3\sqrt{\frac{14}{13}}) = \sqrt{14}$ (max)

condtition 2: $2x-3y=0$

$\frac{45}{2}y^2 + 18y^2 + 5y^2 = 14$

$y = \pm\frac{2}{\sqrt{13}}$

$x = \pm\frac{3}{\sqrt{13}}$

$f(\pm\frac{3}{\sqrt{13}}, \pm\frac{2}{\sqrt{13}}) = 1$ (min)

score: Find f(x) gets 3 pts. Using Lagrange equation gets 3 pts. Didin't find the answer of Lagrange equation loses 2 pts separately.(There are two lines or λ you will get.) Answer is 1 pt separately.

4. (12%) 設 $f(x,y)$ 是雙變數 x, y 之函數。令 $g(t) = f(1+2t, 3+4t)$. 求 $g'(0)$ 及 $g''(0)$。(用 $\frac{\partial f}{\partial x}(1,3)$, $\frac{\partial f}{\partial y}(1,3)$, $\frac{\partial^2 f}{\partial x^2}(1,3)$, $\frac{\partial^2 f}{\partial x \partial y}(1,3)$, $\frac{\partial^2 f}{\partial y^2}(1,3)$ 表出)

Solution:

令 $x = 1+2t$、$y = 3+4t$。

由鏈鎖律得到

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2\frac{\partial f}{\partial x}(x,y) + 4\frac{\partial f}{\partial y}(x,y)$$

再次微分得到

$$\begin{aligned} g''(t) &= 2\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt}\right] + 4\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}\right] \\ &= 4\frac{\partial^2 f}{\partial x^2}(x,y) + 16\frac{\partial^2 f}{\partial x \partial y}(x,y) + 16\frac{\partial^2 f}{\partial y^2}(x,y) \end{aligned}$$

將 $t = 0$ 代入得到

$$\begin{aligned} g'(0) &= 2\frac{\partial f}{\partial x}(1,3) + 4\frac{\partial f}{\partial y}(1,3) \\ g''(0) &= 4\frac{\partial^2 f}{\partial x^2}(1,3) + 16\frac{\partial^2 f}{\partial x \partial y}(1,3) + 16\frac{\partial^2 f}{\partial y^2}(1,3) \end{aligned}$$

評分標準 各階微分，列出鏈鎖律公式得 2 分，微分正確得 2 分，代入 $t = 0$ 算出正確答案得 2 分。

5. (12%) 求過曲面 $xe^{yz} + \ln(y^3 z^2) = \tan^{-1}\left(\frac{x}{z}\right)$ 上點 $(0,1,1)$ 的切面方程式。

Solution:

令 $f(x,y,z) = xe^{yz} + \ln(y^3 x^2) - \tan^{-1}\left(\frac{x}{z}\right)$。

偏微分得到

$$\begin{aligned} f_x(x,y,z) &= e^{yz} - \frac{\frac{1}{z}}{1+\left(\frac{x}{z}\right)^2} \\ f_y(x,y,z) &= xze^{yz} + \frac{3y^2z^2}{y^3z^2} \\ f_z(x,y,z) &= xye^{yz} + \frac{2y^3 z}{y^3 z^2} - \frac{-\frac{x}{z^2}}{1+\left(\frac{x}{z}\right)^2} \end{aligned}$$

代入 $(0,1,1)$ 得到

$$f_x(0,1,1) = e-1, \quad f_y(0,1,1) = 3, \quad f_z(0,1,1) = 2$$

因此切面為

$$(e-1)(x-0) + 3(y-1) + 2(z-1) = 0$$

化簡得到

$$(e-1)x + 3y + 2z = 5$$

評分標準 偏導數正確各得 2 分，代入正確各得 1 分，列出切面公式得 2 分，切面正確得 1 分。

6. (13%) 設 $f(x,y) = \left(\frac{x^2}{4} + y^2 - 1\right)^2 + x^2y^2$

(a) 求在點 $(2,1)$ 的梯度 ∇f。

(b) 求在點 $(2,1)$ 沿著向量 $(3,1)$ 之方向的方向導數。

(c) 在點 $(2,1)$，函數 $f(x,y)$ 增加最快的方向為何(以單位向量表示)？並求沿該方向之方向導數。

(d) 求通過點 $(2,1)$ 之等高線在點 $(2,1)$ 處之切線方程式。

Solution:

(a)

In order to fine the gradient of f at $(2,1)$, we need to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ first.

$$\begin{aligned}\nabla f(2,1) &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\Big|_{(2,1)} \\ &= \Big(2(\frac{x^2}{4}+y^2-1)\frac{x}{2}+2xy^2,\ 2(\frac{x^2}{4}+y^2-1)2y+2x^2y\Big)\Big|_{(2,1)} \\ &= (6,12)\end{aligned}$$

(b)

Directional derivative of f along $(3,1)$ at $(2,1)$ is the inner product of $\nabla f(2,1)$ and $\overrightarrow{u}$ where $\overrightarrow{u}$ is unit vector parallel to $(3,1)$

$$\frac{\partial f}{\partial \overrightarrow{u}} = \nabla f(2,1)\cdot\overrightarrow{u} = (6,\ 12)\cdot(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}) = 3\sqrt{10}$$

(c)

From the definition of gradient, we know the direction that will make f increasing most rapidly is parallel to the gradient of f. Hence the answer to the first question is the unit vector parallel to $(6,\ 12)$ which is $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$

The directional derivative of f along $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ at $(2,1)$ is

$$(6,\ 12)\cdot(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = 6\sqrt{5}$$

(d)

Since the normal vector of tangent line of contour line at $(2,1)$ is $\nabla f(2,1)$, its equation is

$$6(x-2)+12(y-1)=0$$

GRADING CRITERIA

(a) There are 3 points in this subproblem. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ correctly get 1 point respectively, compute the final answer correctly get 1 point

(b) There are 3 points in this subproblem. Find unit vector correctly get 1 point, know how to do inner product get 1 point, answer correct get 1 point.

(c) There are 4 points in this subproblem. Find the direction in the form of unit vector get 2 point, find directional derivative correctly get 2 points. You'll loss 1 point with each computation error.

(d) There are 3 points in this subproblem. Find the normal vector of tangent line get 2 points, answer correct get 1 point.

7. (13%) 找出函數 $f(x,y)=e^{y^2-x^2}(y^2+x^2)$ 的候選點，並決定它是局部極大值，局部極小值或是鞍點。

Solution:

$f(x,y)=e^{y^2-x^2}(y^2+x^2)$

(2分)$f_x=-2x\cdot e^{y^2-x^2}(y^2+x^2)+e^{y^2-x^2}\cdot 2x=e^{y^2-x^2}2x(1-x^2-y^2)$

(2分)$f_y=2y\cdot e^{y^2-x^2}(y^2+x^2)+e^{y^2-x^2}\cdot 2y=e^{y^2-x^2}2y(1+x^2+y^2)$

要算候選點,即為求$\nabla f(x,y)=(f_x,f_y)=(0,0)$之(x,y)解,

(3分)若$f_x=0$, 則$x=0$ 或 $x^2+y^2=1$; 若$f_y=0$, 由於$1+x^2+y^2>1\neq 0$, 勢必$y=0$. 綜合上面取交集,得到候選點有三點:$(0,0),(1,0)$,及$(-1,0)$.

候選點可以使用$D(x,y)=f_{xx}f_{yy}-f_{xy}^2$判斷,因此我們需要計算二次偏微分:

(1分)$f_{xx}=-2x\cdot f_x+e^{y^2-x^2}\{2(1-x^2-y^2)+2x(-2x)\}$

$=e^{y^2-x^2}(4x^4+4x^2y^2-10x^2-2y^2+2)$

(1分)$f_{yy}=2y\cdot f_y+e^{y^2-x^2}\{2(1+x^2+y^2)+2y(2y)\}$

$=e^{y^2-x^2}(4y^4+4x^2y^2+2x^2+10y^2+2)$

(1分)$f_{xy}=f_{yx}=2y\cdot f_x+e^{y^2-x^2}\{2x(-2y)\}=e^{y^2-x^2}(-4xy^3-4x^3y)$

(3分)把候選點們代入,得到$D(0,0)=2*2-0*0=4>0$且$f_{xx}(0,0)=2>0$; $D(1,0)=D(-1,0)=-4e^{-1}*4e^{-1}-0*0=-16e^{-2}<0$.

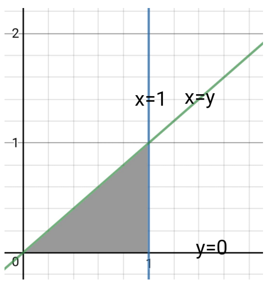
故在$(0,0)$發生局部極小值,而$(\pm 1,0)$皆為鞍點.

1.偏微分有計算錯誤扣一分,幾乎沒積出形式不給分.
2.候選點如果三個都有寫出來但是還多寫不管幾個都只扣一分;如過有漏寫幾個,僅扣少寫幾個的分數.
3.如果沒算出來候選點,卻有寫如何判別且全部正確,1分.
4.D(x,y)的每個點的計算錯誤,頂多扣一分.

8. (12%) 求 $\iint_\Omega \frac{\sin x}{x}dA$，其中 Ω 為 x 軸和 $y=x$, $x=1$ 所圍區域。

Solution:

(3分)把邊界畫出來,灰色部分為所圍出的區域.



(3分)因此根據Fubini的定理,所求積分可以寫為$\int_0^1\int_0^x \frac{\sin x}{x}dydx$或$\int_0^1\int_y^1 \frac{\sin x}{x}dxdy$.

(3分)後者的積分無法寫出來,因此我們用前者,所求$=\int_0^1 \frac{\sin x}{x}*xdx$

(3分)$=-\cos x]_0^1=1-\cos 1$.

1.圖若沒畫,有寫出正確的範圍不扣分;圖畫錯一個地方扣一分.
2.兩個不同方向的積分有寫出來其中一個都給3分,錯一個積分範圍扣1分,寫反扣1分.
3.最後答案一個+-號寫錯扣1,積分結果錯扣2分,差太多沒分.