1062微甲01-04、06-10班期中考解答和評分標準

1. (15 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Please state the tests which you use.

   (a) (5 points) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln(n!)}{n^3\ln n}$

   (b) (5 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt[3]{n}}-\sin\left(\frac{1}{\sqrt[3]{n}}\right)\right)$

   (c) (5 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^{\frac{n^3-n}{2}}\left(\frac{n+1}{n}\right)^{n^2}$

Solution:

(a) Question $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln(n!)}{n^3\ln n}$

Solution Consider that

$$\frac{\ln(n!)}{n^3\ln n}\le\frac{n\ln n}{n^3\ln n}=\frac{1}{n^2}.$$

And series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent by p-series for $p=2>1$.

Therefore we can apply Limit Comparison Test to determine $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln(n!)}{n^3\ln n}$ is absolutely convergent.

2pts Those who thought $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n!)}{n^3\ln n}$ is divergent for any reason and then prove that $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n!)}{n^3\ln n}$ is convergent by Alternating Series Test with a correct process get 2 points.

Correct

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is going to be convergent by Integral Test.

(b) $\ln(n!)=\ln 1+\ln 2+\cdots+\ln n<\ln n+\ln n+\cdots+\ln n=n\ln n$

(c) $\ln(n!)\le\ln(n^n)=n\ln n$

(d) $n\ln n-n+1=\displaystyle\int_1^n\ln x\,dx<\ln(n!)<\int_1^{n+1}\ln x\,dx=(n+1)\ln(n+1)-n$

(e) Stirling Formula: $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\implies\ln(n!)\sim n\ln n-n+\ln\sqrt{2\pi n}$

Incorrect

(a) Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ leads no conclusion.

(b) L'Hôspital Rule: Differentiate $\ln(n!)$ leads mistakes.

(c) Test for Divergence: The limit of a_n as $n\to\infty$ is zero. So we can not use it to conclude the series is divergent.

(d) Limit Comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ leads ∞ and no conclusion.

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\ne 1=\int_1^{\infty}\frac{1}{x^2}\,dx$

(f) $\displaystyle\lim_{n\to\infty}\frac{\ln 1}{\ln n}+\frac{\ln 2}{\ln n}+\cdots+\frac{\ln n}{\ln n}=0+0+\cdots+1$ is wrong.

(b) Let $a_n=\dfrac{1}{\sqrt[3]{n}}-sin\dfrac{1}{\sqrt[3]{n}}$, for any positive integer n. Then, $a_n\ge 0$.

(For $x\ge 0$, $sin(x)=sin(x)-sin(0)=x\cdot cos(\xi)\le x$, for some $\xi\in(0,x)$, by Mean Value

Theorem. Hence, $\frac{1}{\sqrt[3]{n}} - sin\frac{1}{\sqrt[3]{n}} \geq 0$.)
This problem can be decomposed into two parts

1. Convergence of $\sum_{n=1}^{\infty}(-1)^n a_n$ (2 points)
2. Divergence of $\sum_{n=1}^{\infty} a_n$ (3 points)

Convergence of $\sum_{n=1}^{\infty}(-1)^n a_n$

There are two kinds of grading, depending on what kind of method one used.

1. Directly applying Alternating Series Test:
   - (1 point) a_n is decreasing :
     Let $f(x) = \frac{1}{\sqrt[3]{x}} - sin\frac{1}{\sqrt[3]{x}}$. $f'(x) = \frac{-1}{3}x^{-4/3}(1 - cos(x)) \leq 0$, for $x > 0$. Hence, $f(x)$ is decreasing as x increases (when $x > 0$). Since $a_n = f(n)$ for all n, a_n is decreasing as n increases.
   - (1 point) $\lim_{n\to\infty} a_n = 0$:
     $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(\frac{1}{\sqrt[3]{n}} - sin(\frac{1}{\sqrt[3]{n}})) = \lim_{n\to\infty}\frac{1}{\sqrt[3]{n}} - \lim_{n\to\infty} sin(\frac{1}{\sqrt[3]{n}}) = 0 - 0 = 0$, since $sin(x)$ is continuous with respect to x.

   Hence, by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}(-1)^n(\frac{1}{\sqrt[3]{n}} - sin\frac{1}{\sqrt[3]{n}})$ is convergent.
2. Considering convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$ and $\sum_{n=1}^{\infty}(-1)^n sin(\frac{1}{\sqrt[3]{n}})$:
   - (1 point) Convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$:
     $\frac{1}{\sqrt[3]{n}} \geq 0$, $\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}} = 0$ and $\frac{1}{\sqrt[3]{n}}$ is decreasing as n increases. Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$ is convergent.
   - (1 point) Convergence of $\sum_{n=1}^{\infty}(-1)^n sin(\frac{1}{\sqrt[3]{n}})$:
     $sin(\frac{1}{\sqrt[3]{n}}) \geq 0$, $\lim_{n\to\infty} sin(\frac{1}{\sqrt[3]{n}}) = 0$ and $sin(\frac{1}{\sqrt[3]{n}})$ is decreasing as n increases. Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n sin(\frac{1}{\sqrt[3]{n}})$ is convergent.

Divergence of $\sum_{n=1}^{\infty} a_n$

We will use Comparison Test to demonstrate that $\sum_{n=1}^{\infty} a_n$ is divergent. Note that, if one only use upper bound or negative lower bound of a_n to get the divergence of $\sum_{n=1}^{\infty} a_n$, he/she will get 0 point in this part.

- (2 points) Compare $\sum_{n=1}^{\infty} a_n$ with some appropriate series:

  $\lim_{x\to\infty} \frac{x - sin(x)}{x^3} = \lim_{x\to\infty} \frac{1-cos(x)}{3x^2} = \lim_{x\to\infty} \frac{sin(x)}{6x} = \frac{1}{6} \Rightarrow \lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}} - sin\frac{1}{\sqrt[3]{n}}}{(1/\sqrt[3]{n})^3} = \frac{1}{6} > 0.$

  Therefore, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ both converges or both diverges.

- (1 point) Divergence of the "appropriate" series:

  By p-series test or integral test or comparison test, we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, and hence, so is $\sum_{n=1}^{\infty} a_n$. That is, $\sum_{n=1}^{\infty} (\frac{1}{\sqrt[3]{n}} - sin\frac{1}{\sqrt[3]{n}})$ is divergent.

(c) Let $a_n = (-1)^{\frac{n^3-n}{n}} (\frac{n+1}{n})^{n^2}$, for all positive integer n.

- (2 points) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (\frac{n+1}{n})^n = e$
- (2 points) $e > 1$ (strictly larger than 1)
- (1 point) By Root Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{\frac{n^3-n}{n}} (\frac{n+1}{n})^{n^2}$ is divergent.

(01-02班) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Please state the tests which you use.

(a) (5 points) $\sum_{n=2}^{\infty}(-1)^n \frac{\ln(n!)}{n^3 \ln n}$

(b) (5 points) $\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt[n]{2}-1}{\ln n}$

(c) (5 points) $\sum_{n=1}^{\infty} n^5 \frac{4^n - n^3}{(-5)^n + 3^n}$

Solution:

(a)

(b) Observe that $2^{1/n}-1$ is decreasing to zero, and $\ln(n)$ is increasing to infinity. So

$$\frac{2^{1/n}-1}{\ln(n)}$$

is descreasing to zero. Hence by alternating series test,

$$\sum_{n=2}^{\infty}(-1)^n \frac{2^{1/n}-1}{\ln(n)}$$

is convergent. **(2 %)**

On the other hand, observe that

$$\frac{2^{1/n}-1}{\ln(n)} = \frac{e^{\ln(2)/n}-1}{\ln(n)} \geq \frac{\left(1+\frac{\ln(2)}{n}\right)-1}{\ln(n)} = \frac{\ln(2)}{n\ln(n)}.$$

Since $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ is divergent by integral test, the series

$$\sum_{n=2}^{\infty}\frac{2^{1/n}-1}{\ln(n)}$$

is also divergent by comparison test. **(3 %)**

Thereofore, $\sum_{n=2}^{\infty}(-1)^n \frac{2^{1/n}-1}{\ln(n)}$ is conditionally convergent.

(c) Let $a_n = n^5 \frac{4^n-n^3}{(-5)^n+3^n}$. Observe that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{(n+1)^5}{n^5} \times \frac{5^n+(-3)^n}{5^{n+1}+(-3)^{n+1}} \times \frac{4^{n+1}-(n+1)^3}{4^n-n^3}\right) \\
&= \lim_{n\to\infty}\left(\frac{(n+1)^5}{n^5} \times \frac{1+(-3/5)^n}{5+(-3)\times(-3/5)^n} \times \frac{4-(n+1)^3/4^n}{1-n^3/4^n}\right) \\
&= 1 \times \frac{1}{5} \times 4 \\
&= \frac{4}{5}. \ \textbf{(5 \%)}
\end{aligned}$$

Hence by ratio test, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2. (10 points) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{n(\ln n)^{\frac{3}{4}}}$.

Solution:

Write $a_n = \frac{(2x-1)^n}{n(\ln n)^{3/4}}$, by ratio text, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|(2x-1)\frac{n+1}{n}\left(\frac{\ln(n+1)}{\ln n}\right)^{3/4}\right| = |2x-1|$$

Hence, we have $|2x-1| < 1$, or $\left|x - \frac{1}{2}\right| < \frac{1}{2}$**(4%)**.

Now we check the convergence of endpoints:

- When $x = 0$:
  Now $a_n = \frac{(-1)^n}{n(\ln n)^{3/4}}$. Note that $\lim_{n\to\infty} \frac{1}{n(\ln n)^{3/4}} = 0$**(1%)** and $\frac{1}{n(\ln n)^{3/4}}$ is obviously decreasing.**(1%)** By Leibnitz test, it is convergent.**(1%)**
- When $x = 1$:
  Now $a_n = \frac{1}{n(\ln n)^{3/4}}$. Write $f(x) = \frac{1}{x(\ln x)^{3/4}}$, then f is obviously positive (for $x > 1$), decreasing**(1%)**, and continuous. By integral test, we have

  $$\int_2^\infty \frac{\mathrm{d}x}{x(\ln x)^{3/4}} = 4(\ln x)^{1/4}\Big|_2^\infty = \infty\textbf{(1\%)}$$

  Therefore, it is divergent.**(1%)**

  Hence, the radius of convergence is $\frac{1}{2}$ and the convergence interval is $[0, 1)$ □

(01-02班)

(a) (5 points) Find the constant p such that $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{n^p}$ is a finite nonzero constant.

(b) (5 points) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(1-3x)^n}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}$.

Solution:

(a) Let $L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{n^p}$. Observe that

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\ \mathrm{d}x \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \le 1+\int_1^{n} \frac{1}{\sqrt{x}}\ \mathrm{d}x. \ \textbf{(3 \%)}$$

So

$$\frac{2\sqrt{n+1}-2}{n^p} \le \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{n^p} \le \frac{2\sqrt{n}-1}{n^p}.$$

Hence by squeeze theorem,

$$\begin{cases} \text{If } p > 1/2, \text{ then } L = 0, \\ \text{If } p = 1/2, \text{ then } L = 2, \\ \text{If } p < 1/2, \text{ then } L = \infty. \end{cases}$$

Thus the constant $p = 1/2$. **(2 %)**

(b) Let $f_n(x) = \frac{1}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}(1-3x)^n$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right| &= \left(\lim_{n\to\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}}}\right)|1-3x| \\
&= \left(\lim_{n\to\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{\sqrt{n}} \times \frac{\sqrt{n+1}}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}}} \times \frac{\sqrt{n}}{\sqrt{n+1}}\right)|1-3x| \\
&= 2 \times \frac{1}{2} \times 1 \times |1-3x| \\
&= |1-3x|.
\end{aligned}$$

So by ratio test, if $|1-3x| < 1$, or equiveltnly, $0 < x < \frac{2}{3}$, then $\sum_{n=1}^{\infty} f_n(x)$ is convergent **(3 %)**, and if $x < 0$, or $\frac{2}{3} < x$, then $\sum_{n=1}^{\infty} f_n(x)$ is divergent. Now we are going to check $x = 0$ and $x = \frac{2}{3}$.

Observe that $\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{1}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}$. By (a), since

$$\lim_{n\to\infty}\left|\frac{1}{\sqrt{n}}\middle/\frac{1}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}\right| = 2,$$

$\sum_{n=1}^{\infty} f_n(0)$ is divergent by limit comparison test. **(1 %)**

Observe that $\sum_{n=1}^{\infty} f_n(\frac{2}{3}) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}}$. So it is clear that $\sum_{n=1}^{\infty} f_n(\frac{2}{3})$ is convergent by alternating series test. **(1 %)**

Therefore, the interval of convergence of $\sum_{n=1}^{\infty} f_n(x)$ is $(0, \frac{2}{3}]$.

3. (10 points) Let $F(x) = \int_0^x \ln\left(1 + \frac{t^2}{2}\right) dt$.

(a) (6 points) Find the Maclaurin series of $F(x)$ and its radius of convergence.

(b) (4 points) Estimate $F(10^{-1})$ up to an error within 10^{-7}.

Solution:

(a)

Let $F(x) = \int_0^x ln(1 + \frac{t^2}{2})dt$ and by Fundamental Theorem of Calculus we have $F'(x) = ln(1 + \frac{x^2}{2})$.

Since $ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$, substitute $\frac{x^2}{2}$ with x and we have $ln(1+\frac{x^2}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\frac{x^2}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{2^n n}$

$F(x)$ is integrate $F'(x)$ term by term $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n+1}}{2^n n(2n+1)}$

Next, use ratio test to find radius of convergence.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n x^{2n+3}}{2^{n+1}(n+1)(2n+3)} \cdot \frac{2^n n(2n+1)}{(-1)^{n-1}x^{2n+1}}\right| = \left|\frac{x^2}{2}\right|$$

In order to make this alternative series converge, $\left|\frac{x^2}{2}\right|$ need to be less than 1. We have $\left|x\right| < \sqrt{2}$, hence the radius of convergence is $\sqrt{2}$

(b)

Suppose $b_n = \frac{(\frac{1}{10})^{2n+1}}{2^n n(2n+1)}$ and $M_n = \sum_{n=1}^{n} \frac{(-1)^{n-1}(\frac{1}{10})^{2n+1}}{2^n n(2n+1)}$, we know that the error of $M_n(x)$ is bounded by b_{n+1}. Note that

$$b_1 = \frac{(\frac{1}{10})^{-3}}{2 \cdot 1 \cdot 3} = \frac{1}{6000} = 0.000167 > 10^{-7}$$
$$b_2 = \frac{(\frac{1}{10})^{-5}}{4 \cdot 2 \cdot 5} = \frac{1}{4000000} = 2.5 \times 10^{-5} > 10^{-7}$$
$$b_3 = \frac{(\frac{1}{10})^{-7}}{8 \cdot 3 \cdot 7} = \frac{1}{168 \times 10^7} = 5.952 \times 10^{-10} < 10^{-7}$$

Hence the summation of first two term of $F(10^{-1})$ is sufficient to make the error less than 10^{-7}.

$$F(10^{-1}) \approx \frac{1}{6000} - \frac{1}{4000000} \approx 0.0001664$$

GRADING CRITERIA

(a) Finding the Maclaurin series and radius of convergence are 3 points respectively. Write down the basic formula of Maclaurin series will get 1 point, answer correct will get 2 points.

For radius of convergence, use ratio test to find the answer will get 1 point, answer correct will get 2 points.
You will loss 1 point for each calculation error.

(b) If you try to estimate the error $F(10^{-1})$, you will get 2 points even if the final answer is wrong. If the answer is correct, you will get another 2 points.
You will loss 1 point for each calculation error.

4. (8 points)

(a) (4 points) Identify the power series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1}$ as an elementary function.

(b) (4 points) Find the sum $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \frac{1}{189\sqrt{3}} + \cdots$

Solution:

(a)

Method 1.

$\because \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ (2%)

$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} = \tan^{-1} 2x$ (2%)

Method 2.

for $|x| < \frac{1}{2}$

$\because (\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1})' = 2 \cdot \sum_{n=0}^{\infty} (-1)^n (2x)^{2n} = \frac{2}{1+4x^2}$ (2%)

$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} = \int \frac{2}{1+4x^2} dx = \tan^{-1} 2x + C$ (1%)

The series equals 0 when $x = 0$, so $C = 0$. (1%)

Therefore, $\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} = \tan^{-1} 2x$

(b)

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \frac{1}{189\sqrt{3}} + \ldots\ldots$

$= \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \ldots\ldots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\frac{1}{\sqrt{3}})^{2n+1}$ (2%)

$= \tan^{-1} \frac{1}{\sqrt{3}}$ (1%)

$= \frac{\pi}{6}$ (1%)

5. (12 points) Let $\mathbf{r}(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le \pi$, be a vector function that parametrizes a curve in space.

(a) (3 points) Find the arc length of the curve.

(b) (6 points) At what point on the curve is the osculating plane parallel to the plane $x+\sqrt{3}y-z = 0$?

(c) (3 points) Find the curvature of the curve.

Solution:

(a) $\mathbf{r}'(t) = (\cos t - \cos t + t\sin t)\mathbf{i} + (-\sin t + \sin t + t\cos t)\mathbf{j} + 2t\mathbf{k} = t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}$

$\Rightarrow |\mathbf{r}'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + (2t)^2} = \sqrt{5}t$ (2 points)

$\therefore$ Arc length $L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}t\,dt = \frac{\sqrt{5}}{2}\pi^2$ (1 point)

(b) Osculating plane is spanned by the tangent and normal vector of the curve $\mathbf{r}(t)$, so we need to find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$\because \mathbf{r}'(t) = (t\sin t, t\cos t, 2t)$ and $|\mathbf{r}'(t)| = \sqrt{5}t \Rightarrow \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}}(\sin t, \cos t, 2)$

$\therefore \mathbf{T}'(t) = \frac{1}{\sqrt{5}}(\cos t, -\sin t, 0) \Rightarrow \mathbf{N}(t) = (\cos t, -\sin t, 0)$

Normal vector of osculating plane $\vec{n} = (1, \sqrt{3}, -1)$ parallel to $\mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}}(2\sin t, 2\cos t, -1)$

$\Rightarrow t = \frac{\pi}{6}$, that is, the osculating plane at $\left(\frac{1}{2} - \frac{\sqrt{3}\pi}{12}, \frac{\sqrt{3}}{2} + \frac{\pi}{12}, \frac{\pi^2}{36}\right)$ is parallel to the plane $x + \sqrt{3}y + z = 0$. (2 points for each $\mathbf{T}, \mathbf{T}'$, 1 point for each $\mathbf{N}$, Point)

(c) (Method I)By (b) we have $|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}$, (1 point)

Curvature $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \frac{1}{5t}$ (2 points)

(Method II) By (a) we have $\mathbf{r}'(t) = t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}$

$\Rightarrow \mathbf{r}''(t) = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + 2\mathbf{k}$ (1 point)

$\therefore \mathbf{r}'(t) \times \mathbf{r}''(t) = 2t^2\sin t\mathbf{i} + 2t^2\cos t\mathbf{j} - t^2\mathbf{k} \Rightarrow |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{5}t^2$ (1 point)

$\Rightarrow \kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{5}t^2}{\left(\sqrt{5}t\right)^3} = \frac{1}{5t}$ (1 point)

(sol II)

(i) $\mathbf{r}'(t) = t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{5}t$

(ii) $\mathbf{r}''(t) = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + 2\mathbf{k}$

(iii) $\mathbf{r}'(t)\times\mathbf{r}''(t) = 2t^2\sin t\mathbf{i} + 2t^2\cos t\mathbf{j} - t^2\mathbf{k} \Rightarrow |\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{5}t^2$

(iv) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}}(\sin t, \cos t, 2)$

(v) $\mathbf{T}'(t) = \dfrac{1}{\sqrt{5}}(\cos t, -\sin t, 0) \Rightarrow |\mathbf{T}'(t)| = \dfrac{1}{\sqrt{5}}$

(vi) $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = (\cos t, -\sin t, 0)$

(vii) $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \dfrac{1}{\sqrt{5}}(2\sin t, 2\cos t, -1)$

(a) Arc length $L = \displaystyle\int_0^{\pi} |\mathbf{r}'(t)|\, dt = \int_0^{\pi} \sqrt{5}t dt = \frac{\sqrt{5}}{2}\pi^2$

(b) Normal vector of osculating plane $\vec{n} = (1, \sqrt{3}, -1)$ parallel to $\mathbf{B}(t) = \dfrac{1}{\sqrt{5}}(2\sin t, 2\cos t, -1)$ $\Rightarrow t = \dfrac{\pi}{6}$, that is, the osculating plane at $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}\pi}{12}, \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{12}, \dfrac{\pi^2}{36}\right)$ is parallel to the plane $x + \sqrt{3}y + z = 0$.

(c) $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \dfrac{1}{5t}$ or $\kappa = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{5}t^2}{\left(\sqrt{5}t\right)^3} = \dfrac{1}{5t}$

6. (10 points) Let surface S be given by $S = \{(x,y,z) \in \mathbb{R}^3 | \sin(xyz) = x + 2y + 3z\}$.

(a) (4 points) On the surface, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial y}{\partial x}$.

(b) (2 points) Find an equation of the tangent plane to the surface S at $(2,-1,0)$.

(c) (4 points) Suppose, when restricted to the surface S, a differentiable function f attains a local maximum value at the point $(2,-1,0)$ with $f(2,-1,0) = 10$ and $f_x(2,-1,0) = 2$. Let (x_0, y_0, z_0) be a point which is close to the point $(2,-1,0)$ and lies on another surface $\sin(xyz) = z + 2y + 3z + 10^{-2}$. Use the linear approximation to estimate $f(x_0, y_0, z_0)$.

Solution:

Define $g(x,y,z) = \sin(xyz) - x - 2y - 3z$.

(a) Treating z implicitly as a function of x and y, by chain rule we can differentiate the equation $g(x,y,z) = 0$ as follows:

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} = 0.$$

We obtain

$$\frac{\partial z}{\partial x} = -\frac{g_x}{g_z} = -\frac{yz\cos(xyz) - 1}{xy\cos(xyz) - 3}.$$

Similarly,

$$\frac{\partial y}{\partial x} = -\frac{g_x}{g_y} = -\frac{yz\cos(xyz) - 1}{xz\cos(xyz) - 2}.$$

(b) The tangent plane to the surface S at $(2,-1,0)$ is

$$\nabla g(2,-1,0) \cdot < x-2, y-(-1), z-0 > = -(x-2) - 2(y+1) - 5z = 0, \quad \text{or} \quad x + 2y + 5z = 0.$$

(c) Since $f(2,-1,0)$ is a local maximum value, by the method of Lagrange multiplier, there is a number λ such that $\nabla f(2,-1,0) = \lambda \nabla g(2,-1,0)$.

From the x-exponent of the equation and the fact that $f_x(x,y,z) = 2$ we find that $\lambda = -2$ and thus $\nabla f(2,-1,0) = -2\nabla g(2,-1,0)$. It follows from the linear approximation of g at the point $(2,-1,0)$ that

$$10^{-2} = g(x_0,y_0,z_0) - g(2,-1,0) \approx \nabla g(2,-1,0) \cdot < x_0 - 2, y_0 + 1, z_0 >$$

Therefore, the linear approximation of f at $(2,-1,0)$ yields

$$\begin{aligned} f(x_0,y_0,z_0) &\approx f(2,-1,0) + \nabla f(2,-1,0) \cdot < x_0 - 2, y_0 + 1, z_0 > \\ &= 10 - 2\,\nabla g(2,-1,0) \cdot < x_0 - 2, y_0 + 1, z_0 > \approx 10 - 2 \cdot 10^{-2} = 9.98. \end{aligned}$$

Marking Scheme

(a) 1 point for each derivation using chain rule or direct use of formula;
1 point for each correct answer.

(b) 1 point for the formula of the tangent plane and 1 point for the correct equation.

(c) 1 point for using Lagrange's method; 0.5 point for the correct λ.
1 point for each approximation of f and g; 0.5 point for the correct estimate.

7. (13 points) Let $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) (3 points) Is $f(x,y)$ continuous at $(0,0)$? Justify your answer.

(b) (2 points) Find the gradient vector $\nabla f(0,0)$.

(c) (4 points) Is $f_x(x,y)$ continuous at $(0,0)$? Justify your answer.

(d) (4 points) Find the maximum and minimum directional derivatives of f at the point $(0,0)$ among the directions of all the unit vectors $\mathbf{u}$.

Solution:

(a) $x = r\cos\theta,\ y = r\sin\theta$

$$|f(x,y)| = \left|\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2}\right|$$
$$= r\,|\cos^3\theta+\sin^3\theta|$$
$$\leq r\,|\cos^3\theta| + r\,|\sin^3\theta| \leq 2r$$

So, $f(x,y) \to 0 = f(0,0)$ as $r \to 0$ is as $(x,y) \to (0,0)$

Therefore, f is continuous at (0,0).

Grading Policy:
(1) 3 points for correct proof.
(2) No partical points.

(b) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)}{h} = \lim_{h\to 0} \dfrac{1}{h}\dfrac{h^3}{h^2} = 1$

$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h)}{h} = \lim_{h\to 0} \dfrac{1}{h}\dfrac{h^3}{h^2} = 1$

$\nabla f(0,0) = \vec{i} + \vec{j}$

Grading Policy:
(1)Correct limits for 1 point.
(2)Correct answer for 1 point.

(c)Away from $(0,0)$,

$$f_x = \frac{3x^2(x^2+y^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2}$$
$$= \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$$

Let $x = r\cos\theta, y = r\sin\theta$

$f_x = \cos^4\theta + 3\cos^2\theta\sin^2\theta - 2\cos\theta\sin^3\theta$

f_x is $D_{\theta=0}$.

$\theta = 0 \to f(x,y) = 1$, and $\theta = \dfrac{\pi}{2} \to f(x,y) = 1$

$f_x(x,y)$ is NOT continuous at (0,0).

Other method:

$$f_x = \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$$

Let $y = mx \to f_x = \dfrac{2m^2}{1+m^2}$

So, the limit as $(x,y) \to (0,0)$ along the different lines $y = mx$ is different for different m. $f_x(x,y)$ is NOT continuous.

Grading Policy:
(1)Correct f_x for 2 points.
(2)Correct limit at $(0,0)$ for 2 points.

(d) $\vec{u} = (\cos\theta, \sin\theta)$
$D_u f(0,0) = \lim_{r\to 0} \frac{f(r\cos\theta, r\sin\theta)}{r}$
$= \lim_{r\to 0} \frac{\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2}}{r}$
$= \cos^3\theta + \sin^3\theta$

Let $g(\theta) = \cos^3\theta + \sin^3\theta$
$\to g'(\theta) = -3\sin\theta\cos^2\theta + 3\cos\theta\sin^2\theta$
If $g'(\theta) = 0$, then $\theta = 0$ or $\frac{\pi}{4}$ or $\frac{\pi}{2}$ or $\frac{5\pi}{4}$ and so on.
Maximum is $g(0) = g(\frac{\pi}{2}) = 1$.
Minimum is $g(\pi) = g(\frac{3\pi}{2}) = -1$.

Grading Policy:
(1)Find the directional derivative in direction of $\vec{u} = (\cos\theta, \sin\theta)$ for 2 points.
(2)Correct maximum and minimum argunents for 1 point.
(3)Correct answer for 1 point.

8. (12 points) Let $f(x,y) = 1 + 3x^2 - 2x^3 + 3y - y^3$.

(a) (6 points) Find the local maximum and minimum values and saddle point(s) of $f(x,y)$.

(b) (6 points) Find the extreme values of $f(x,y)$ on the region D bounded by the triangle with vertices $(-2,2)$, $(2,2)$ and $(2,-2)$.

Solution:

(a)

$f_x = 6x - 6x^2 = 6x(1-x)$

$f_y = 3 - 3y^2 = 3(1-y)(1+y)$

solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ imply critical points are :$(1,1),(1,-1),(0,1),(0,-1)$

$f_{xy} = f_{yx} = 0$

$f_{xx} = 6 - 12x$

$f_{yy} = -6y \qquad D = f_{xx}f_{yy} = 72xy - 36y^2$

at point $(1,1)$

$D(1,1) = 36 > 0, f_{xx}(1,1) = -6 < 0$,local maximum$f(1,1) = 4$

at point $(1,-1)$

$D(1,-1) = -36 < 0$,saddle point

at point $(0,1)$

$D(0,1) = -36 < 0$,saddle point

at point $(0,-1)$

$D(0,-1) = 36 > 0, f_{xx}(0,-1) = 6 > 0$,local minimum$f(0,-1) = -1$

(b)

for y=2,$-2 \le x \le 2$

$f(x,2) = -2x^3 + 3x^2 - 1$ denote $g_1(x)$

$g_1'(x) = -6x(x-1)$,solve $g_1'(x) = 0, x = 0, 1$

$f(-2,2) = 27, f(0,2) = -1, f(1,2) = 0, f(2,2) = -5$

for x=2,$-2 \le y \le 2$

$f(2,y) = -y^3 + 3y - 3$ denote $g_2(y)$

$g_2'(y) = -3(y-1)(y+1)$,solve $g_2'(y) = 0, y = -1, 1$

$f(2,-2) = -1, f(2,-1) = -5, f(2,1) = -1, f(2,2) = -5$

for x+y=0,$-2 \le x \le 2$

$f(x,-x) = -x^3 + 3x^2 - 3x + 1$ denote $g_3(x)$

$g_3'(x) = -3(x-1)^2$,solve $g_3'(x) = 0, x = 1$

$f(-2,2) = 27, f(1,-1) = 0, f(2,-2) = -1$

Comparing above point and critical points we get ,$\begin{cases} maximum = 27, at\,(-2,2) \\ minimum = -5, at\,(2,-1),(2,2) \end{cases}$

[Grading]

(a)

(2 points)

$f_x, f_y, f_{xx}, f_{yy}$and find 4 critical points (4 points)

correct determine each critical point is location maximum,location minimum,saddle point

if you don't write the local maximum and local minimum values, you will lose 1 point

(b)

if you only consider points in the interior of the triangle and extremum on boundary, you will get at most 4 points

if you only consider points in the interior of the triangle and corners of the triangle, you will get at most 3 points

if you consider all points but doesn't dissusion, you will get at most 5 points
if you only consider points in the interior of the triangle, you will get at most 1 point

9. (10 points) By the Extreme Value Theorem, a continuous function on a sphere attains both absolute maximum and minimum values. Find the extreme values of $f(x,y,z) = \ln(x+2)+\ln(y+2)+\ln(z+2)$ on the sphere $x^2+y^2+z^2 = 3$.

Solution:

Step1.

Let $g(x,y,z) = x^2+y^2+z^2 = 3$.

According to the method of Lagrange multipliers, we solve the equation $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 3$. This gives

$$\frac{1}{x+2} = \lambda \cdot 2x \qquad \frac{1}{y+2} = \lambda \cdot 2y \qquad \frac{1}{z+2} = \lambda \cdot 2z \qquad x^2+y^2+z^2 = 3$$

(3pts)

Step2.

Note that $\lambda \neq 0$ because $\lambda = 0$ implies $\frac{1}{x+2} = 0$, which is impossible.
Thus we have

$$\frac{1}{\lambda} = 2x(x+2) = 2y(y+2) = 2z(z+2)$$

From $2x(x+2) = 2y(y+2)$, we have

$$0 = x^2 - y^2 + 2x - 2y = (x-y)(x+y+2)$$

which gives

$$y = x \qquad or \qquad y = -x-2$$

Similarly, from $2x(x+2) = 2z(z+2)$, we have

$$z = x \qquad or \qquad z = -x-2$$

Case1. $y = x$ and $z = x$
From $x^2+y^2+z^2 = 3$, we have $3x^2 = 3$ and then $x = 1,-1$.
Thus we have two points $(1,1,1)$, $(-1,-1,-1)$.

Case2. $y = x$ and $z = -x-2$
From $x^2+y^2+z^2 = 3$, we have $3x^2+4x+4 = 3$ and then $x = -\frac{1}{3},-1$.
Thus we have two points $(-\frac{1}{3},-\frac{1}{3},-\frac{5}{3})$, $(-1,-1,-1)$.

Case3. $y = -x-2$ and $z = x$
From $x^2+y^2+z^2 = 3$, we have $3x^2+4x+4 = 3$ and then $x = -\frac{1}{3},-1$.
Thus we have two points $(-\frac{1}{3},-\frac{5}{3},-\frac{1}{3})$, $(-1,-1,-1)$.

Case4. $y = -x - 2$ and $z = -x - 2$
From $x^2 + y^2 + z^2 = 3$, we have $3x^2 + 8x + 8 = 3$ and then $x = -\frac{5}{3}, -1$.
Thus we have two points $(-\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3})$, $(-1, -1, -1)$.

Hence f has possible extreme values at the points $(1, 1, 1)$, $(-1, -1, -1)$, $(-\frac{1}{3}, -\frac{1}{3}, -\frac{5}{3})$, $(-\frac{1}{3}, -\frac{5}{3}, -\frac{1}{3})$ and $(-\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3})$. **(5pts)**

Step3.

We compare the values of $f(x, y, z)$ at these points:

- $f(1, 1, 1) = \ln 27$
- $f(-1, -1, -1) = 0$
- $f(-\frac{1}{3}, -\frac{1}{3}, -\frac{5}{3}) = f(-\frac{1}{3}, -\frac{5}{3}, -\frac{1}{3}) = f(-\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3}) = \ln \frac{25}{27}$

Therefore the maximum value of f on the sphere $x^2 + y^2 + z^2 = 3$ is $\ln 27$
and the minimum value is $\ln \frac{25}{27}$. **(2pts)**