1052微甲06-10班期中考解答和評分標準

1. (15% total)

   (a) (5%) Derive the MacLaurin series of $\tan^{-1} x$.

   (b) (5%) Find the value of $a \in \mathbb{R}$ such that the limit $\lim_{x \to 0} \dfrac{\sin(ax) - \sin x - \tan^{-1} x}{x^3}$ is finite.

   (c) (5%) Evaluate the above limit.

Solution:

(a) $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$ (1%)

$$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int \sum_{j=0}^{\infty} (-1)^j x^{2j} dx = \sum_{j=0}^{\infty} \int (-1)^j x^{2j} dx = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1} + c \text{ (3\%)}$$

(P.S. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$ by calculate $f'(0)$, ..., $f^{(5)}(0)$ (2%)).
Let $x = 0$, $c = \tan^{-1} 0 = 0$. Its radius of convergence is 1. $|x| < 1$ (1%)

(b) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$ (3%) (P.S. $\sin x = x - \dfrac{x^3}{3!} + \ldots$ (2%))
$\dfrac{\sin(ax) - \sin x - \tan^{-1} x}{x^3} = (a-2)\dfrac{1}{x^2} + \{\dfrac{1-a^3}{3!} + \dfrac{1}{3}\} + O(x^2)$.
Thus, $a = 2$. (2%)

(c) The limit is $\dfrac{1}{6}(1-8) + \dfrac{1}{3} = \dfrac{-5}{6}$ (5%)

2. (10% total) Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)\ln(n+1)}$.

(a) (5%) Determine its radius of convergence.

(b) (5%) Determine its interval of convergence.

Solution:

(a) Let $c_n = \frac{(-1)^n}{(n+1)\ln(n+1)}$. Use *Ratio Test* we have

$$\lim_{n\to\infty} \frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} = |x| \lim_{n\to\infty} \frac{n+1}{n+2} \frac{\ln(n+1)}{\ln(n+2)} = |x| \cdot 1 < 1$$

Where $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} \overset{L}{=} \lim_{n\to\infty} \frac{n+2}{n+1} = 1$.
Thus the radius of convergence $R = 1$.

(b) Let $b_n = |c_n|$.
Try $x = 1$. (i) b_n clearly decreasing (ii) $\lim_{n\to\infty} b_n = 0$ obviously.
Thus by *Alternating Series Test* $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.
Try $x = -1$. (i) b_n positive (ii) b_n decreasing (iii) $f(n) = b_n$ continuous
Then by *Integral Test*, $\int_1^{\infty} f(x)dx = \ln\ln(x+1)|_1^{\infty}$ diverges implies $\sum_1^{\infty} b_n$ diverges.
Therefore the interval of convergence is $(-1, 1]$.

Grading

(a) (2 pts) State correct test.
(2 pts) Correct calculation.
(1 pt) Correct answer

(b) (1 pt) Case $x = 1$, state correct test.
(1 pt) Case $x = 1$, correct calculation
(1 pt) Case $x = -1$, state correct test.
(1 pt) Case $x = -1$, correct calculation
(1 pt) Correct answer

Remarks

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum b_n$ converges implies $\sum a_n$ converges, but $\sum b_n$ diverges means nothing.

3. (20% total) Consider the space curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2t^3}{3}\mathbf{k}$.

(a) (5%) Find the arc length of the curve from $t = 0$ to $t = a$.

(b) (5%) Find the curvature $\kappa(0)$ at $t = 0$.

(c) (5%) Find the unit tangent $\mathbf{T}(0)$ at $t = 0$.

(d) (5%) Find the unit normal $\mathbf{N}(0)$ at $t = 0$.

Solution:

(a)

$$r'(t) = (1, 2t, 2t^2) \quad (2\%)$$

$$|r'(t)| = \sqrt{1 + 4t^2 + 4t^4} = 2t^2 + 1 \quad (1\%)$$

$$if \quad a \geq 0$$

$$s = \int_0^a |r'(t)|dt = \int_0^a (2t^2 + 1)dt = \frac{2}{3}a^3 + a \quad (2\%)$$

$$if \quad a < 0$$

$$s = \int_a^0 |r'(t)|dt = \int_a^0 (2t^2 + 1)dt = -\frac{2}{3}a^3 - a$$

(b)

(Method I)

$$\frac{dT}{ds} = \kappa N \Rightarrow \kappa = \frac{|\frac{dT}{dt}|}{|r'(t)|} \quad (1\%)$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = (\frac{1}{2t^2 + 1}, \frac{2t}{2t^2 + 1}, 1 - \frac{1}{2t^2 + 1})$$

$$T'(t) = (\frac{-4t}{(2t^2 + 1)^2}, \frac{-4t^2 + 2}{(2t^2 + 1)^2}, \frac{4t}{(2t^2 + 1)^2}) \quad (2\%)$$

$$\kappa(0) = \frac{|T'(0)|}{|r'(0)|} = \frac{|(0, 2, 0)|}{|(1, 0, 0)|} = 2 \quad (2\%)$$

(Method II)

$$\kappa = \frac{|r' \times r''|}{|r'|^3} \quad (1\%)$$

$$r'(t) = (1, 2t, 2t^2) \quad r'(0) = (1, 0, 0)$$

$$r''(t) = (0, 2, 4t) \quad r''(0) = (0, 2, 0)$$

$$|r'(0) \times r''(0)| = |(0, 0, 2)| = 2 \quad (2\%)$$

$$|r'(0)|^3 = |(1, 0, 0)|^3 = 1$$

$$\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = 2 \quad (2\%)$$

沒有代入t=0扣2%

(c)

$$T(0) = \frac{r'(0)}{|r'(0)|} = (1, 0, 0) \quad (5\%)$$

若因前面$r'(t), |r'(t)|$算錯而算錯$T(0)$會酌量扣分

沒有代入t=0扣2%

(d)

$$N(0) \parallel T'(0) = (0, 2, 0) \quad (3\%)$$

若因前面$r'(t), |r'(t)|, T'(t)$算錯而算錯$T'(0)$會酌量扣分

$$N(0) = \frac{(0,2,0)}{|(0,2,0)|} = (0,1,0) \quad (2\%)$$

沒有代入t=0扣2%

4. (11%) Let $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$.

(a) (6%) Express $\dfrac{\partial z}{\partial x}$ in terms of r, θ and partial derivatives with respect r, θ.

(b) (5%) Express $\dfrac{\partial^2 z}{\partial x^2}$ in terms of r, θ and partial derivatives with respect r, θ.

Solution:

(a). Note that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

$$\begin{aligned}
&\frac{\partial z}{\partial x} \\
&= \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \quad (\clubsuit\ 3\%) \\
&= \frac{\partial z}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\cdot\frac{-y}{x^2+y^2} \quad (\clubsuit\ 2\%) \\
&= \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\cdot\frac{-\sin\theta}{r} \quad (\clubsuit\ 1\%)
\end{aligned}$$

(b). From above, we know $\dfrac{\partial}{\partial x} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}$.

$$\begin{aligned}
&\frac{\partial^2 z}{\partial x^2} \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\cdot\frac{-\sin\theta}{r}\right) \quad (\clubsuit\ 3\%) \\
&= \frac{\partial^2 z}{\partial r^2}\cos^2\theta - \frac{\partial^2 z}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial z}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\sin\theta}{r}\left(-\frac{\partial^2 z}{\partial r\partial\theta}\cos\theta + \frac{\partial z}{\partial r}\sin\theta + \frac{\partial^2 z}{\partial\theta^2}\frac{\sin\theta}{r} + \frac{\partial z}{\partial\theta}\frac{\cos\theta}{r}\right) \\
&= \frac{\partial^2 z}{\partial r^2}\cos^2\theta - \frac{\partial^2 z}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta}{r} + \frac{\partial^2 z}{\partial\theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial z}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial z}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2} \quad (\clubsuit\ 2\%)
\end{aligned}$$

5. (20% total) Let $f(x,y,z) = (x^2+z^2)\sin\frac{\pi xy}{2} + yz^2$ and a point $\mathbf{p} = (1,1,-1)$. Answer the following:

(a) (5%) Find the gradient of f at $\mathbf{p}$.

(b) (5%) Find the approximate value of $f(0.98, 1.02, -0.97)$.

(c) (5%) Find the plane tangent to the level surface through $\mathbf{p}$ defined by $f(x,y,z) = f(\mathbf{p}) = 3$.

(d) (5%) If a bird flies through $\mathbf{p}$ directly to the point $(2,-1,1)$ with speed 5, what is the rate of change of f as seen by the bird at $\mathbf{p}$?

Solution:

(a)

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,-1)} = \left.\left[2x\sin\frac{\pi xy}{2} + (x^2+z^2)\left(\cos\frac{\pi xy}{2}\right)\frac{\pi y}{2}\right]\right|_{(1,1,-1)} = 2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,-1)} = \left.\left[(x^2+z^2)\left(\cos\frac{\pi xy}{2}\right)\frac{\pi x}{2} + z^2\right]\right|_{(1,1,-1)} = 1$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,-1)} = \left.\left[2z\sin\frac{\pi xy}{2} + 2yz\right]\right|_{(1,1,-1)} = -4$$

$$\therefore \nabla f(1,1,-1) = \left.\left[\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right]\right|_{(1,1,-1)} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

Remark: (4%) for the partial derivatives; (1%) for evaluation of the gradient

(b) The linear approximation $L(x,y,z)$ of $f(x,y,z)$ at the point $\mathbf{p} = (1,1,-1)$ is

$$L(x,y,z) = f(1,1,-1) + \left.\frac{\partial f}{\partial x}\right|_{(1,1,-1)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,1,-1)}(y-1) + \left.\frac{\partial f}{\partial z}\right|_{(1,1,-1)}(z+1)$$

$$= 3 + 2(x-1) + (y-1) - 4(z+1) \qquad (3\%)$$

$$\therefore f(0.98, 1.02, -0.97) \approx L(0.98, 1.02, -0.97) = 3 + 2(-0.02) + 0.02 - 4(0.03) = 2.86 \qquad (2\%)$$

(c) Notice that $f(\mathbf{p}) = f(1,1,-1) = 3$, which means that the plane tangent to the level surface is the tangent plane of $f(x,y,z)$ at the point $\mathbf{p}$. Therefore, the tangent plane equation is

$$2(x-1) + (y-1) - 4(z+1) = 0 \qquad (5\%)$$

(d) The unit vector $\mathbf{u}$ from point $\mathbf{p} = (1,1,-1)$ to point $\mathbf{q} = (2,-1,1)$ is

$$\mathbf{u} = \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \qquad (1\%)$$

Then, the directional derivative

$$D_{\mathbf{u}}f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{u} = (2,1,-4)\cdot\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = -\frac{8}{3} \qquad (3\%)$$

The rate of change of f as seen by the bird at $\mathbf{p}$ with speed $v = 5$ is

$$D_{\mathbf{u}}f(\mathbf{p}) \times v = -\frac{40}{3} \qquad (1\%)$$

6. (12%) Find the local extreme values and saddle points of $f(x,y) = x^2y - xy^2 + xy - y^2$.

Solution:

$$\begin{cases} f_x = & 2xy - y^2 + y & = & y(2x - y + 1) = 0 & \cdots(*) \\ f_y = & x^2 - 2xy + x - 2y & = & (x+1)(x-2y) = 0 & \cdots(\dagger) \end{cases}$$

For $(*)$:

i. If $y = 0$, then $x(x+1) = 0$ from $(\dagger)$ $\Rightarrow x = 0$ or -1.
   Hence, we have $(0,0), (-1,0)$.

ii. If $y = 2x+1$, then $3x^2 + 5x + 2 = 0$ from $(\dagger)$ $\Rightarrow x = -1$ or $-\dfrac{2}{3}$.
   Hence, we have $(-1,-1), (-\dfrac{2}{3}, -\dfrac{1}{3})$.

Note that we can get the same result by considering $(\dagger)$. Therefore, the critical points are $\boxed{(0,0), (-1,0), (-1,-1), (-\frac{2}{3}, -\frac{1}{3})}$.
(1% for each point)
Since

$$f_{xx} = 2y, \quad f_{xy} = f_{yx} = 2x - 2y + 1, \quad f_{yy} = -2x - 2,$$

we now have

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -4(x+1)y - (2x - 2y + 1)^2.$$

i. $D(0,0) = -1 < 0$ $\Rightarrow$ $\boxed{(0,0) \text{ is a saddle point.}}$ (2%)

ii. $D(-1,0) = -1 < 0$ $\Rightarrow$ $\boxed{(-1,0) \text{ is a saddle point.}}$ (2%)

iii. $D(-1,-1) = -1 < 0$ $\Rightarrow$ $\boxed{(-1,-1) \text{ is a saddle point.}}$ (2%)

iv. $D(-\dfrac{2}{3}, -\dfrac{1}{3}) = \dfrac{1}{3} > 0, f_{xx}(-\dfrac{2}{3}, -\dfrac{1}{3}) = -\dfrac{2}{3} < 0$
   $\Rightarrow$ $\boxed{f(x,y) \text{ has a local maximum at} (-\frac{2}{3}, -\frac{1}{3}).}$ (1%)
   And the local maximum value at $(-\dfrac{2}{3}, -\dfrac{1}{3})$ is $f(-\dfrac{2}{3}, -\dfrac{1}{3}) = \boxed{\frac{1}{27}}$. (1%)

7. (12%) Find the maximum and the minimum of the function $f(x, y) = 3x^2 - 2y^2$ on the curve $2x^2 - 2xy + y^2 = 1$.

Solution:

Let $g(x, y) = 2x^2 - 2xy + y^2 - 1 = 0$ By applying the method of Lagrange multipliers, we need to solve

$$\nabla f = \lambda \nabla g \quad \text{[2 points]} \quad \text{and } g(x, y) = 0$$

$$\text{or } \begin{cases} 6x = \lambda(4x - 2y) & (1) \\ -4y = \lambda(-2x + 2y) & (2) \\ 2x^2 - 2xy + y^2 - 1 = 0 & (3) \end{cases} \quad \begin{array}{l} \text{[6 points]} \\ \text{(1 points per coefficient in (1)(2))} \end{array}$$

Clearly, $x \neq 0$, $y \neq 0$, $\lambda \neq 0$, or $g(x, y)$ fails to be 0. So, dividing (1) by (2) gives

$$\frac{3x}{-2y} = \frac{2x - y}{-x + y} \Rightarrow 3x^2 - 7xy + 2y^2 = 0$$

$$(3x - y)(x - 2y) = 0 \quad \therefore 3x = y \text{ or } x = 2y$$

- Case 1: $3x = y$. Plug this into (3) can get
  $x^2 = 1/5, y^2 = 9/5 \quad \therefore f(x, y) = 3x^2 - 2y^2 = \frac{3}{5} - \frac{18}{5} = -3$
- Case 2: $x = 2y$. Plug this into (3) can get
  $y^2 = 1/5, x^2 = 4/5 \quad \therefore f(x, y) = 3x^2 - 2y^2 = \frac{12}{5} - \frac{2}{5} = 2$

Since the extreme value must exist, 2 is the absolute maximum [2 points]
and -3 is the absolute minimum [2 points]