1. (10%) Let $p \in (0, 1)$. A sequence $\{x_n\}_{n=1}^{\infty}$ is given by

$$x_1 = \sqrt{p} \quad \text{and} \quad x_{n+1} = \sqrt{p + x_n}, \text{ for } n \geq 1.$$

Determine whether the sequence is convergent or divergent with an argument. If it is convergent, find the limit.

Solution:

• Claim 1: $\{x_n\}$ is bounded (4 points)
Prove that $0 < x_n < 2 \; \forall n \in \mathbb{N}$:
Base case: $0 < x_1 = \sqrt{p} < \sqrt{1} < 2$.
Assume that $0 < x_k < 2$ for $k \geq 1$, we have $0 < x_{k+1} = \sqrt{p + x_k} < \sqrt{1+2} < 2$, thus the claim is proved by mathematical induction.

• Claim 2: $\{x_n\}$ is increasing (4 points)
Base case: $x_2 = \sqrt{p + x_1} = \sqrt{p + \sqrt{p}} > \sqrt{p} = x_1$.
Assume that $x_k > x_{k-1}$ for $k \geq 2$, we have $x_{k+1} = \sqrt{p + x_k} > \sqrt{p + x_{k-1}} = x_k$, thus the claim is proved by mathematical induction.
By Claim 1 and 2, $\{x_n\}$ is monotonic (increasing) and bounded (above), thus it converges by the Monotonic Sequence Theorem.

• Find the limit: (2 points)
Since the sequence converges, assume that $\lim_{n \to \infty} x_n = L$, then
$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sqrt{p + x_n}$
$\Rightarrow L = \sqrt{p + L}$
$\Rightarrow L^2 - L - p = 0$
$\Rightarrow L = \dfrac{1 \pm \sqrt{1+4p}}{2}$, take $L = \dfrac{1 + \sqrt{1+4p}}{2}$ since $0 < L < 2$.

2. (12%)

(a) (5%) Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{n}{(1+n^3)^p}$ is convergent.

(b) (7%) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution:

(a) (5%)
(method 1)
Let $b_n = \frac{n}{n^{3p}} = \frac{1}{n^{3p-1}}$. $a_n, b_n > 0$ for all $n > 0$. For all $p \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{(1+n^3)^p}}{\frac{n}{n^{3p}}} = \lim_{n\to\infty} (\frac{n^3}{1+n^3})^p = \lim_{n\to\infty} (\frac{1}{\frac{1}{n^3}+1})^p = 1 \textbf{ (3\%)}$$

so both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent or divergent **(1%)** by the Limit Comparison Test. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3p-1}}$ is convergent if and only if $3p-1 > 1$, which implies $p > \frac{2}{3}$, we know that $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $p > \frac{2}{3}$. **(1%)**
(method 2)
(1). $p \le 0$: $\lim_{n\to\infty} \frac{n}{(1+n^3)^p} = \infty$, so it is divergent by the Limit Divergence Test. **(1%)**
(2). $0 < p \le \frac{2}{3}$: $0 < \frac{n}{(n^3+n^3)^{\frac{2}{3}}} \le \frac{n}{(n^3+n^3)^p} \le \frac{n}{(1+n^3)^p}$ for all $n > 0$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{n}{(n^3+n^3)^{\frac{2}{3}}} = \sum_{n=1}^{\infty} 2^{-\frac{2}{3}} \frac{1}{n}$ is divergent, so $\sum_{n=1}^{\infty} \frac{n}{(1+n^3)^p}$ is also divergent when $0 < p \le \frac{2}{3}$. **(2%)**
(3). $p > \frac{2}{3}$: $0 < \frac{n}{(1+n^3)^p} \le \frac{1}{n^{3p-1}}$ for all $n > 0$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^{3p-1}}$ is convergent when $p > \frac{2}{3}$, $\sum_{n=1}^{\infty} \frac{n}{(1+n^3)^p}$ is also convergent when $p > \frac{2}{3}$. **(2%)**
By (1)(2)(3), $\sum_{n=1}^{\infty} \frac{n}{(1+n^3)^p}$ is convergent if and only if when $p > \frac{2}{3}$.

(b) (7%)
First, we show that $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n}$ is **NOT** absolutelt convergent.
Let $a_n = \frac{\tan^{-1} n}{n}$, and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} > 0$.
By limit comparison test, we know $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \tan^{-1} n}{n} \right| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. **(3%)**
Next, we show that $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n}$ is convergent.
In order to apply the alternating series test, we need to show that the sequence $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$
(1)$\{a_n\}$ is decreasing (at least for $n \ge N$): **(3%)**
Let $f(x) = \tan^{-1} x$ for $x \in [1, \infty)$, then $f'(x) = \frac{\frac{x}{1+x^2} - \tan^{-1} x}{x^2}$, we need to show that $f'(x) < 0$:
(**method 1**):
$\left(\frac{x}{1+x^2} - \tan^{-1} x\right)' = \frac{1-x^2}{(1+x^2)^2} - \frac{1}{1+x^2} = \frac{-2x^2}{(1+x^2)^2} \le 0$ and $\left(\frac{x}{1+x^2} - \tan^{-1} x\right)\Big|_{x=0} = 0$, so $f'(x) < 0$.
(**method 2**):

$\lim_{x\to\infty} \frac{x}{1+x^2} = 0$ and $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$,.
So there exsits $N \in \mathbb{N}$ such that $\frac{n}{1+n^2} - \tan^{-1} n < 0$ for $n \geq N$
(2) $\lim_{n\to\infty} a_n = 0$: **(1%)**
(**method 1**):
$\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = \lim_{n\to\infty} \tan^{-1} n \cdot \lim_{n\to\infty} \frac{1}{n} = \frac{\pi}{2} \cdot 0 = 0$
(**method 2**):
Note that $0 \leq \frac{\tan^{-1} n}{n} \leq \frac{\pi/2}{n}$ for $n \geq 1$. Since $\lim_{n\to\infty} \frac{\pi/2}{n} = 0$, we have $\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0$
So, by alternating series test, we know that $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n}$ is convergent.
And therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n}$ is conditionally convergent.

3. (12%) A plane curve C is parameterized by $\mathbf{r}(t) = (\cos t + t\sin t, \sin t - t\cos t), t > 0$, as Figure 1.

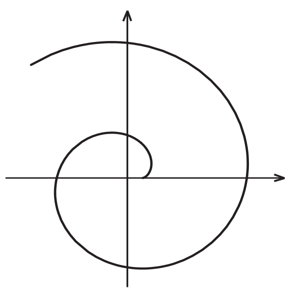

Figure 1: The plane curve C.

(a) Compute the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$.

(b) Show that all centers of osculating circles, $\mathbf{r}(t) + \dfrac{1}{\kappa(t)}\mathbf{N}(t)$, lie on a circle.

Solution:

(a) (10%)
(Method 1)

$$r'(t) = (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t) = (t\cos t, t\sin t)\ \textbf{(1\%)},\ \text{so}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\ \textbf{(2\%)} = (\cos t, \sin t)\ \textbf{(1\%)}.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}\ \textbf{(2\%)} = (-\sin t, \cos t)\ \textbf{(1\%)}.$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}\ \textbf{(2\%)} = \frac{1}{t}\ \textbf{(1\%)}.$$

(Method 2)
The plane curve can be view as $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 0\mathbf{k}$. Thus, we have

$$\mathbf{r}'(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}''(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}''(t) = 0\mathbf{i} + 0\mathbf{j} + t^2\mathbf{k}\ \textbf{(1\%)}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\ \textbf{(2\%)} = \cos t\mathbf{i} + \sin t\mathbf{j} + 0\mathbf{k}\ \textbf{(1\%)},\ \text{so}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}\ \textbf{(2\%)} = -\sin t\mathbf{i} + \cos t\mathbf{j} + 0\mathbf{k}\ \textbf{(1\%)}.$$

$$\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\ \textbf{(2\%)} = \frac{1}{t}\ \textbf{(1\%)}.$$

(b) (2%)

$$\mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = (\cos t + t\sin t, \sin t - t\cos t) + t(-\sin t, \cos t) = (\cos t, \sin t).$$

Thus, all centers of osculating circles lies on a circle. **(2%)**

4. (10%) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + 2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ or explain why the limit does not exist.

(b) Compute the directional derivative $D_{\mathbf{u}} f(0,0)$, where $\mathbf{u} = (\cos\theta, \sin\theta)$ is any direction.

Solution:

(a) (5 points)
If we evaluate the limit along the curves $y = mx^2$,

$$\lim_{\substack{(x,\,y)\,\to\,(0,\,0) \\ y\,=\,mx^2}} f(x,y) = \lim_{x\to 0} \frac{x^2 \cdot mx^2}{x^4 + 2m^2x^4} = \frac{m}{1+2m^2}$$

which varies as the value of m varies. Thus the limit does not exist.

(b) (5 points in total)
Note that since $\lim_{(x,y)\to(0,0)} f$ does not exist, f is not continuous at $(0,0)$ and in turn f is not differentiable at $(0,0)$. Thus the relation $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ can NOT be used here.

By definition, when $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ (already unit length),

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \lim_{h\to 0} \frac{f(0 + h\cos\theta, 0 + h\sin\theta) - f(0,0)}{h} \quad \text{(2 points)} \\ &= \lim_{h\to 0} \frac{\frac{h^3 \cos^2\theta \sin\theta}{h^4\cos^4\theta + 2h^2\sin^2\theta} - 0}{h} \\ &= \lim_{h\to 0} \frac{\cos^2\theta \sin\theta}{h^2\cos^4\theta + 2\sin^2\theta} \\ &= \begin{cases} \dfrac{\cos^2\theta}{2\sin\theta}, & \sin\theta \neq 0 \ (\theta \neq n\pi, n \in \mathbf{Z};\ \text{2 points}) \\ 0, & \sin\theta = 0 \ (\theta = n\pi, n \in \mathbf{Z};\ \text{1 point}) \end{cases} \end{aligned}$$

(Note: if you tempted to use $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ and calculated $f_x(0,0) = 0$ and $f_y(0,0) = 0$ correctly **by definition**, 2 points will be credited.)

5. (10%) A differentiable function $f(x, y)$ has the following properties:

- $f(0, 0) = 1$.
- $D_{\mathbf{u}} f(0, 0) = 2$, where $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$.
- $D_{\mathbf{v}} f(0, 0) = \frac{3}{\sqrt{2}}$, where $\mathbf{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(a) What is the maximal rate of increase of $f(x, y)$ at $(0, 0)$?

(b) Use the linearization of $f(x, y)$ at $(0, 0)$ to estimate $f(0.07, -0.05)$.

Solution:

(a) Let $\nabla f(0, 0) = (a, b)$

then we have $\frac{3}{5}a + \frac{4}{5}b = 2$ and $\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b = \frac{3}{\sqrt{2}}$

$\Rightarrow \nabla f(0, 0) = (2, 1)$ (3 pts)

Maximum rate of change $= |(2, 1)| = \sqrt{5}$ (2 pts)

(b) $L(x, y) = f(0, 0) + f_x(0, 0) \cdot (x - 0) + f_y(0, 0) \cdot (y - 0)$

$= 1 + 2x + y$ (4 pts)

$L(0.07, -0.05) = 1 + 0.14 - 0.05 = 1.09$ (1 pts)

6. (10%) Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x.$$

Solution:

First, we find all critical points by calculating $\vec{\nabla} f(x,y) = 0$.

i.e., $\begin{cases} f_x &= 3x^2 + 6x - 9 = 0\textbf{(2\%)} \\ f_y &= -3y^2 + 6y = 0\textbf{(2\%)} \end{cases}$

Solving the equation, we get four points: $P_1 = (1,0)$, $P_2 = (1,2)$, $P_3 = (-3,0)$, and $P_4 = (-3,2)$.

Next, we compute the Hessian matrix of f : $\text{Hess}(f) = \begin{pmatrix} 6x+6 & 0 \\ 0 & -6y+6 \end{pmatrix}$ **(2%)**

At P_1, we have $D(P_1) = 72 > 0$, and $f_{xx}(P_1) = 12 > 0$, so P_1 is a local minimum with $f(1,0) = -5$ **(1%)**

At P_2, we have $D(P_2) = -72 < 0$, so P_2 is a saddle point **(1%)**

At P_3, we have $D(P_3) = -72 < 0$, so P_2 is a saddle point **(1%)**

At P_4, we have $D(P_4) = 72 > 0$, and $f_{xx}(P_4) = -12 < 0$, so P_4 is a local maximum with $f(-3,2) = 31$ **(1%)**

7. (14%) *Viviani's curve*, sometimes also called *Viviani's window*, is the intersection of the cylinder $(x-1)^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$, as Figure 2.

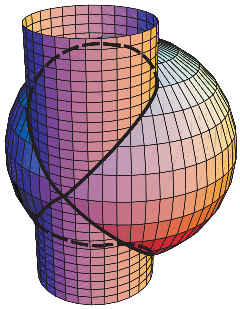

Figure 2: Viviani's curve.

(a) Find the tangent line equation of the Viviani's curve at $P(1,1,\sqrt{2})$.

(b) Find the points on the Viviani's curve that are nearest to and farthest from $Q(2,0,2)$.

Solution:

(a) Let $F_1(x,y,z)=(x-1)^2+y^2-1=0$ and $F_2(x,y,z)=x^2+y^2+z^2-4=0$.
We compute

$$\nabla F_1(x,y,z)=(2(x-1),2y,0)\Rightarrow \nabla F_1(1,1,\sqrt{2})//(0,1,0)\ \text{(1pt)}$$
$$\nabla F_2(x,y,z)=(2x,2y,2z)\Rightarrow \nabla F_2(1,1,\sqrt{2})//(1,1,\sqrt{2})\ \text{(1pt)}$$

The directional vector of the tangent line is

$$(0,1,0)\times(1,1,\sqrt{2})=(\sqrt{2},0,-1).\ \text{(2pts)}$$

So the tangent line equation is

$$\begin{cases} x(t) &= 1+\sqrt{2}t \\ y(t) &= 1 \\ z(t) &= \sqrt{2}-t \end{cases},t\in\mathbb{R}\quad \text{(1pt)}$$

(b) Consider the Lagrange function $L(x,y,z,\lambda,\mu)=(x-2)^2+y^2+(z-2)^2-\lambda((x-1)^2+y^2-1)-\mu(x^2+y^2+z^2-4)$.
Then we will find all critical points of L :

$$\begin{cases} L_x &= 2(x-2)-2\lambda(x-1)-2\mu x=0\Rightarrow(1-\lambda-\mu)x=2-\lambda \\ L_y &= 2y-2\lambda y-2\mu y=0\Rightarrow(1-\lambda-\mu)y=0 \\ L_z &= 2(z-2)-2\mu z=0\Rightarrow z(1-\mu)=2 \\ L_\lambda &= -((x-1)^2+y^2-1)=0\Rightarrow(x-1)^2+y^2=1 \\ L_\mu &= -(x^2+y^2+z^2-4)=0\Rightarrow x^2+y^2+z^2=4. \end{cases}\quad \text{(3pts)}$$

(A) If $y=0$, then $x=0$ or $x=2$, and it implies $(x,y,z)=(0,0,2)$ and $(0,0,-2)$. (Remark that $(2,0,0)$ does not satisfies $L_z=0$.) The distance will be 2 and $2\sqrt{5}$, respectively. (2pts)

(B) If $\lambda+\mu=1$, then $\lambda=2$, $\mu=-1$, and it gives $z=1$ and then $x=\frac{3}{2}$ and $y=\pm\frac{\sqrt{3}}{2}$. The distance will be $\sqrt{2}$. (2pts)

Hence the nearest points are $(\frac{3}{2},\pm\frac{\sqrt{3}}{2},1)$. (1pt) The farthest points is $(0.0,-2)$. (1pt)

8. (15%) Let $f(x) = \displaystyle\int_{-1}^{x} \frac{1}{\sqrt{t^2+2t+2}}\,dt$.

(a) Find the Taylor series for $f(x)$ centered at $a = -1$. (Hint: Complete the square first.)

(b) Find $f^{(9)}(-1)$ and $f^{(10)}(-1)$.

(c) Write down the 3rd-degree Taylor polynomial $T_3(x)$ for $f(x)$ centered at $a = -1$, and calculate $T_3\left(-\frac{1}{2}\right)$. Estimate the error $\left|f\left(-\frac{1}{2}\right) - T_3\left(-\frac{1}{2}\right)\right|$ by some estimation theorem.

Solution:

(a) Note that $x^2 + 2x + 1 = (x+1)^2 + 1$.

$$
\begin{aligned}
f'(x) &= \frac{1}{\sqrt{x^2+2x+2}} = \left(1 + (x+1)^2\right)^{-\frac{1}{2}} \\
&= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left((x+1)^2\right)^n && \left|(x+1)^2\right| < 1 \\
&= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (x+1)^{2n} && |x+1| < 1 \\
f(x) &= c_0 + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(x+1)^{2n+1}}{2n+1} && |x+1| < 1 \\
f(-1) &= c_0 + 0 = \int_{-1}^{-1} \frac{1}{\sqrt{t^2+2t+2}}\,dt = 0
\end{aligned}
$$

Thus $c_0 = 0$ and

$$
f(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(x+1)^{2n+1}}{2n+1} \qquad |x+1| < 1
$$

It is also OK to write $\binom{-\frac{1}{2}}{n} = \dfrac{(-1)^n(2n!)}{4^n(n!)^2}$

Although not grading, one can find the interval of convergence is $[-2, 0]$. The proof is at the last part.

(b)

$$
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!}(x+1)^k = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{2n+1}(x+1)^{2n+1}
$$

By comparing the coefficient, we have $k = 9 = 2n+1, n = 4$ and then

$$
f^{(9)}(-1) = 9! \cdot \binom{-\frac{1}{2}}{4} \frac{1}{2\cdot 4+1} = 8! \cdot \frac{\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2}}{4\cdot 3\cdot 2\cdot 1} = (1\cdot 3\cdot 5\cdot 7)^2 = 11025
$$

and $k = 10 = 2n+1, n \notin \mathbb{N}$ thus $f^{(10)}(-1) = 0$

(c) Take the terms until power 3,

$$
\begin{aligned}
T_3(x) &= 0 + \binom{-\frac{1}{2}}{0} \frac{(x+1)^1}{1} + 0 + \binom{-\frac{1}{2}}{1} \frac{(x+1)^3}{3} \\
&= (x+1) - \frac{1}{6}(x+1)^3 \\
T_3\left(-\frac{1}{2}\right) &= \left(\frac{1}{2}\right) - \frac{1}{6}\left(\frac{1}{2}\right)^3 = \frac{23}{48}
\end{aligned}
$$

Let $R(x) = f(x) - T_3(x)$. We want to find an estimate to $\left|R\left(-\frac{1}{2}\right)\right|$.

- (Method 1) Use *Taylor's Inequality*.
  If $\left|f^{(4)}(x)\right| \le M$ for $|x+1| \le \frac{1}{2}$, then $|R_3(x)| \le \frac{M}{4!}|x+1|^4$.
  Thus $\left|R\left(-\frac{1}{2}\right)\right| = \left|R_3\left(-\frac{1}{2}\right)\right| \le \frac{M}{4!}\left|-\frac{1}{2}+1\right|^4 = \frac{M}{384}$.

Now we are going to try to find some M satisfies $\left|f^{(4)}(x)\right| \leq M$ for ALL $|x+1| < \frac{1}{2}$. (Not only $x=-1$ or $-\frac{1}{2}$. There is a remark later.)

$$\begin{aligned}
f^{(2)}(x) &= \frac{-1}{2}\left(1+(x+1)^2\right)^{\frac{-3}{2}} \cdot 2(x+1) \\
f^{(3)}(x) &= \frac{3}{4}\left(1+(x+1)^2\right)^{\frac{-5}{2}} \cdot 4(x+1)^2 + \frac{-1}{2}\left(1+(x+1)^2\right)^{\frac{-3}{2}} \cdot 2 \\
f^{(4)}(x) &= \frac{-15}{8}\left(1+(x+1)^2\right)^{\frac{-7}{2}} \cdot 8(x+1)^3 + \frac{3}{4}\left(1+(x+1)^2\right)^{\frac{-5}{2}} \cdot (8+4)(x+1) \\
&= \left(1+(x+1)^2\right)^{\frac{-7}{2}}\left(-15(x+1)^3 + 9(x+1)\left(1+(x+1)^2\right)\right) \\
&= \frac{3(x+1)\left(3-2(x+1)^2\right)}{\sqrt{\left(1+(x+1)^2\right)^7}}
\end{aligned}$$

Note that $0 \leq |x+1| \leq \frac{1}{2}$ and

$$\left|f^{(4)}(x)\right| \leq \frac{3 \cdot \frac{1}{2} \cdot (3 - 2 \cdot 0)}{\sqrt{\left(1+0^2\right)^7}} = \frac{9}{2}$$

Take $M = \frac{9}{2}$ and $\left|R\left(-\frac{1}{2}\right)\right| \leq \frac{3}{256}$

- (Method 2) Use *Alternating Series Estimation Theorem.* Let

$$b_n = (-1)^n \binom{-\frac{1}{2}}{n} \frac{1}{2n+1}\left(-\frac{1}{2}+1\right)^{2n+1} = c_n \frac{1}{(2n+1)} \frac{1}{2^{2n+1}}$$

Where $c_n = (-1)^n \binom{-\frac{1}{2}}{n}$. Then $c_0 = 1$ and $c_n = \frac{2n-1}{2n} c_{n-1}$.
Note that both b_n and c_n are always positive.
Now (i) $c_n \leq c_{n-1}$, thus $\{c_n\}$ is decreasing, so does $\{b_n\}$
(ii) $c_n \leq 1$, thus $0 \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} \frac{1}{(2n+1)} \frac{1}{2^{2n+1}} = 0$
Thus $\sum_{n=0}^{\infty} (-1)^n b_n$ is an alternating series. Therefore

$$\left|R\left(-\frac{1}{2}\right)\right| = |R_1| \leq b_2 = (-1)^2 \binom{-\frac{1}{2}}{2} \cdot \frac{1}{5} \cdot \frac{1}{2^5} = \frac{3}{1280}$$

This method is simpler and 5 times accurate then the previous one.

Grading

(a) Total 5 pts.
(1 pt) Find $f'(x)$.
(2 pts) Find the Taylor series of $f'(x)$.
(1 pt) Write out the Taylor series of $f(x)$.
(1 pt) For integrate coefficient.
Calculating $f(x) = \sinh^{-1}(x+1)$ does not count. Those ONLY calculating the integral without a series get (only) 2 pts.
Radius of convergence does not count, but costs 1 pt if answered incorrect.
Interval of convergence does not count, even answered incorrect.

(b) Total 5 pts.
(3 pts) Find $f^{(9)}(-1) = c_9 \cdot 9!$, $n=4$. Missing 9! costs 2 pts.
(2 pts) Find $f^{(10)}(-1)$.

(c) Total 5 pts.
(1 pt) Find $T_3(x)$.

(1 pt) Find $T_3\left(-\frac{1}{2}\right)$
(1 pt) State out which estimation theorem is used.
(1 pt) State or use the theorem correctly.
(1 pt) Find an suitable upper bound for $|f(-\frac{1}{2}) - T_3(-\frac{1}{2})|$
For Method 1, writing M without a way how to find it costs 1 pt.
For Method 2, not fully checking the criterion of alternating costs 1 pt.
Accurate answer is not available. Use a rational number to estimate.

Remarks

- C_n^m is not good when m is not a nonnegative integer. Use $\binom{m}{n}$ instead.
- The interval of convergence in (a) can be found as following:
  Let $(-1)^n c_n = \binom{-\frac{1}{2}}{n}$, $c_n = (-1)^n\binom{-\frac{1}{2}}{n}$, then
  1. $\{c_n\}$ is nonnegative.
  2. $c_0 = 1$ and $\dfrac{c_n}{c_{n-1}} = \dfrac{2n-1}{2n}$.
  3. We now prove $c_n \leq \dfrac{1}{\sqrt{n+1}}$.
     Clearly case $n = 0$ holds. Now $n \geq 1$, if $c_{n-1} \leq \dfrac{1}{\sqrt{n}}$, then
     $$c_n = \frac{2n-1}{2n}c_{n-1} \leq \frac{2n-1}{2n}\frac{1}{\sqrt{n}} = \frac{\sqrt{(2n-1)^2(n+1)}}{\sqrt{(2n)^2(n)}}\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$$
     Since $(2n-1)^2(n+1) = 4n^3 - 3n + 1 \leq 4n^3 = (2n^2)(n)$.

  Therefore, both $|x+1| = 1$ or -1 cases, we have
  $$\sum_{n=0}^{\infty}\left|\binom{-\frac{1}{2}}{n}\frac{1}{2n+1}\right| \leq \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}\frac{1}{2n+1} \leq \sum_{n=0}^{\infty}\frac{1}{2n\sqrt{n}}$$
  Which is absolute convergent by *p-series test*.
  Then the interval of convergence is $[-2, 0]$.
  The reason choosing $\dfrac{1}{\sqrt{n+1}}$ will be clear if one knows *Stirling's Formula*, which gives an approximation of the factorial $n!$.
- In Method 1 in (c), *Taylor's inequality* needs $|f^{(4)}(x)| \leq M$ for all $|x+1| \leq \frac{1}{2}$. Actually, if $|f^{(4)}(\frac{-1}{2})|$ is the global maximum, then all will be fine. Unfortunately, that is not the case.

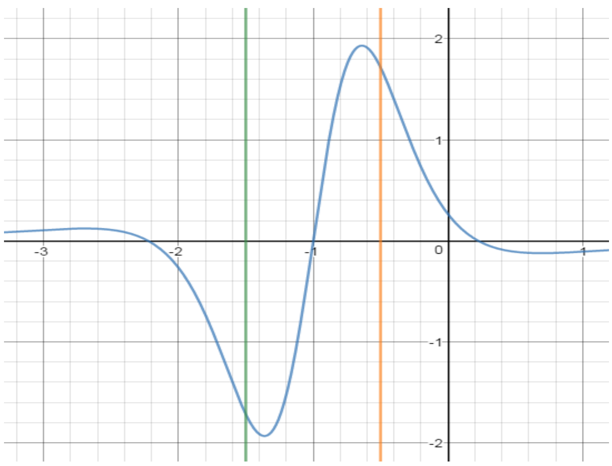

Figure 3: Both $x = -1/2$ (orange) and $x = -3/2$ (green) not global extreme.

- In (c), When estimating *error* of $f(x)$, use "$\leq$" but not "$\approx$". The latter one is really dangerous, since the error of $f(x)$ may be even smaller then the error of approximating, leading to an inaccurate result.

9. (12%) Consider the power series $f(x) = \sum_{n=2}^{\infty} \frac{1}{n(n-1)3^n}(x-2)^n$.

(a) Find the interval of convergence for $f(x)$.

(b) Write down the power series representation for $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ and find its sum in the interior of the interval of convergence.

Solution:

(a) By ratio test $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{|x-2|}{3} < 1$ if $-1 < x < 5$ (4 pts)

At $x = 5$, $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converge. (1 pts)

At $x = -1$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}$ converge. (1 pts)

Thus interval of convergence is $-1 \le x \le 5$

(b) $\frac{d}{dx}(f(x)) = \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{(n-1)}(\frac{x-2}{3})^{n-1}$ (2 pts)

Now, compute $g(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)}(x)^{n-1}$ then $\frac{d}{dx}f(x) = \frac{1}{3}g(\frac{x-2}{3})$

First, $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$\Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ RHS is $g(x)$

Thus $\frac{d}{dx}(f(x)) = -\frac{1}{3}\ln(\frac{5-x}{3})$ (4 pts)