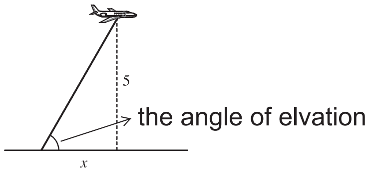
1051微甲06-10班期中考解答和評分標準

1. (10%) A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane travelling at that time?



Solution:

Let the angle of elevation be $\theta(t)$, and the horizontal displacement of the plane from the tracking telescope be $x(t)$, then from the figure we have

$$\tan\theta(t) = \frac{5}{x(t)}, \text{ or equivalently, } x(t) = 5\cot\theta(t) \qquad \text{[3 points]}$$

And, we are given that

$$\frac{d}{dt}\theta(t)\Big|_{\theta=\frac{\pi}{3}} = -\frac{\pi}{6} \qquad \text{[1 points]}$$

Therefore, the velocity of the plane is

$$\begin{aligned}\frac{dx}{dt}\Big|_{\theta=\frac{\pi}{3}} &= 5\cdot\left(-\csc^2\theta\right)\cdot\frac{d\theta}{dt}\Big|_{\theta=\frac{\pi}{3}} \qquad \text{[4 points]}\\ &= 5\cdot\left(-(\frac{2}{\sqrt{3}})^2\right)\cdot\left(-\frac{\pi}{6}\right)\\ &= \frac{10\pi}{9}\,(km/min.) \qquad \text{[2 points]}\end{aligned}$$

(The other way)

$$\begin{aligned}&\tan\theta(t) = \frac{5}{x(t)} \qquad \text{[3 points]}\\ &\Rightarrow \sec^2\theta\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt} \qquad \text{[4 points]}\\ &\Rightarrow 2^2(-\frac{\pi}{6}) = -\frac{5}{(\frac{5}{\sqrt{3}})^2}\frac{dx}{dt}\Big|_{\theta=\frac{\pi}{3}} \qquad \text{[1 points]}\\ &\Rightarrow \frac{dx}{dt}\Big|_{\theta=\frac{\pi}{3}} = \frac{10\pi}{9}\,(km/min.) \qquad \text{[2 points]}\end{aligned}$$

[Grading Criterion]
Write down the equation correctly. [3 points]
Use the given conditions correctly. [1 points]
Differentiate the equation correctly. [4 points]
Calculate the velocity correctly. [2 points]

2. (a) (6%) Find the linear approximation of $\tan^{-1} x$ at the point p.
   (b) (4%) Use (a) to approximate $\tan^{-1} \frac{3}{5}$ with $p = \tan\left(\frac{\pi}{6}\right)$.

Solution:

(a) The linear approximation $L(x)$ of a function $f(x)$ at a point p is given by

$$f(x) \approx L(x) = f(p) + f'(p)(x - p).$$

Let $f(x) = \tan^{-1} x$, then we have $f'(x) = \frac{1}{1+x^2}$.
Therefore, at the point p,

$$\tan^{-1} x \approx \tan^{-1} p + \frac{1}{1+p^2}(x - p). \qquad \text{[6 points]}$$

(b) From part (a), we have

$$\begin{aligned} \tan^{-1} \frac{3}{5} &\approx \tan^{-1} p + \frac{1}{1+p^2}\left(\frac{3}{5} - p\right) \\ &= \tan^{-1}(\tan(\frac{\pi}{6})) + \frac{1}{1+\tan^2(\frac{\pi}{6})}\left(\frac{3}{5} - \tan(\frac{\pi}{6})\right) \\ &= \frac{\pi}{6} + \frac{3}{4}\left(\frac{3}{5} - \frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{9}{20} - \frac{\sqrt{3}}{4} \qquad \text{[4 points]} \end{aligned}$$

[Grading Criterion]
Part (a) Correct answer will get 6 points, otherwise, no point.
Part (b) Correct answer will get 4 points. If the steps are correct but $\tan(\frac{\pi}{6})$ is evaluated wrongly, 2 points.

3. Find the following limits.

(a) (4%) $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)}$ (b) (4%) $\lim_{x\to 0} \frac{(x-1)^{1/3}+(x+1)^{1/3}}{x}$ (c) (4%) $\lim_{x\to 0} \left(\cos(x^2)\right)^{\frac{1}{x^4}}$

Solution:

(a) $-1 \le \sin(\frac{\pi}{x}) \le 1$

$\frac{1}{e} \le e^{\sin(\frac{\pi}{x})} \le e$

$\frac{\sqrt{x}}{e} \le \sqrt{x}e^{\sin(\frac{\pi}{x})} \le \sqrt{x}e$

$\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = 0 = \lim_{x\to 0^+} \sqrt{x}e$

By squeeze theorem

$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\frac{\pi}{x})} = 0$

(b) $\lim_{x\to 0} \frac{\sqrt[3]{x-1}+\sqrt[3]{x+1}}{x}$

$= \lim_{x\to 0} \frac{(x-1)+(x+1)}{x(\sqrt[3]{x-1}^2 - \sqrt[3]{x-1}\sqrt[3]{x+1} + \sqrt[3]{x+1}^2)}$

$= \lim_{x\to 0} \frac{2}{\sqrt[3]{x-1}^2 - \sqrt[3]{x-1}\sqrt[3]{x+1} + \sqrt[3]{x+1}^2}$

$= \frac{2}{3}$

(c) $\lim_{x\to 0} \left(\cos(x^2)\right)^{\frac{1}{x^4}}$

$= e^{\lim_{x\to 0} \frac{\ln(\cos(x^2))}{x^4}}$ $\frac{0}{0}$, use 羅必達

$= e^{\lim_{x\to 0} \frac{\frac{-\sin(x^2)}{\cos(x^2)}2x}{4x^3}}$

$= e^{\lim_{x\to 0} \frac{-\sin(x^2)}{x^2}\frac{1}{2\cos(x^2)}}$

$= e^{\frac{-1}{2}}$

第三題評分標準

(a)

出現$\lim_{x\to 0^+} e^{\sin(\frac{\pi}{x})}$但沒有發現(說明)此極限不存在，其餘概念正確 -1

出現$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\frac{\pi}{x})} = \lim_{x\to 0^+} \sqrt{x} * \lim_{x\to 0^+} e^{\sin(\frac{\pi}{x})}$ 並且沒有用到夾擠的概念，其餘概念正確 -2

把$\lim_{x\to 0^+} e^{\sin(\frac{\pi}{x})}$看成無限大並用羅必達 -2～-4(視後面算式的合理以及完整性而定)

把$\lim_{x\to 0^+} \frac{\sin(\frac{\pi}{x})}{(\frac{\pi}{x})}$ 看成無限大並用羅必達 -2～-4(視後面算式的合理以及完整性而定)

小錯誤 -1

(b)

計算錯誤 -1

上下同乘以錯誤的數字(因子) -1～-3 (視錯誤因子對後面算式的影響而定)

(c)

不影響後面算式前提下的計算錯誤 -1

微分計算錯誤 -2

有取 ln 至少有1分

沒有把取 ln 的結果代回題目要求的式子 -1

PS

若(b)(c)兩題之中至少用了一次羅必達， 但(b)(c)兩題都沒有檢查是否滿足$\frac{0}{0}$的情況，則共計扣1分

4. Suppose that $f(x) = \begin{cases} \sin x + b\ln(x+1) + c \text{ if } x \geq 0 \\ e^{x^2} \text{ if } x < 0 \end{cases}$.

(a) (4%) Find b, c such that $f(x)$ is continuous.

(b) (4%) Find b, c such that $f(x)$ is differentiable.

(c) (4%) For b, c in (b), is $f'(x)$ continuous?

Solution:

(a)

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} [\sin x + b\ln(x+1) + c] = c$ (1pt)

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{x^2} = 1$ (1pt)

$f(0) = c$

Hence, $f(x)$ is continuous at $x = 0 \Leftrightarrow c = 1,\ b \in \mathbb{R}$. (2pt)

(b)

$$\begin{aligned}\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+} \frac{\sin x + b\ln(x+1) + 1 - 1}{x-0} \\ &= \lim_{x\to 0^+} \frac{\sin x + b\ln(x+1)}{x} \\ &= \lim_{x\to 0^+} \left(\frac{\sin x}{x} + b\frac{\ln(x+1)}{x}\right) = 1 + b \text{ (1pt)}\end{aligned}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{e^{x^2}-1}{x-0} \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0^-} \frac{2xe^{x^2}}{1} = 0 \text{ (1pt)}$$

Hence, $f(x)$ is differentiable at $x = 0 \Leftrightarrow c = 1,\ b = -1$ (2pt)

(c)

For $b = -1$ and $c = 1$, $f(x)$ is differentiable everywhere, then

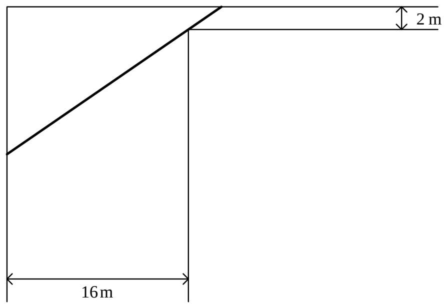
$$f'(x) = \begin{cases} \cos x - \frac{1}{x+1}, & x \geq 0 \\ 2xe^{x^2}, & x < 0 \end{cases}$$

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left[\cos x - \frac{1}{x+1}\right] = 0$ (1pt)

$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2xe^{x^2} = 0$ (1pt), and $f'(0) = 0$ (1pt)

Hence, $f'(x)$ is continuous. (1pt)

5. (15%) A steel pipe is carried down a hallway 16 meter wide. At the end of the hall there is a right angled turn into a narrower hallway 2 meter wide. What is the length of the longest pipe that can be carried horizontally around the corner?



Solution:

Let the width of aisle is $l(\theta)$ where $\theta \in (0, \pi)$ is the angle between the pipe and the horizontal line. Therefore, we can have $l(\theta) = \frac{16}{\cos(\theta)} + \frac{2}{\sin(\theta)}$. We need to find the minimum of the $l(\theta)$; in this way we can find the length of the longest pipe.

$l'(\theta) = 16\sec(\theta)\tan(\theta) - 2\csc(\theta)\cot(\theta)$

To find the minimum of the $l(\theta)$, we should solve $l'(\theta) = 0$.

$l'(\theta) = 16\sec(\theta)\tan(\theta) - 2\csc(\theta)\cot(\theta) = 16\frac{\sin\theta}{\cos^2\theta} - 2\frac{\cos\theta}{\sin^2\theta} = \frac{16\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$

$\Rightarrow\ 16\sin^3\theta - 2\cos^3\theta = 0 \Rightarrow \tan^3\theta = \frac{1}{8} \Rightarrow \tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2}$(10 points)

We can check that $l''(\tan^{-1}\frac{1}{2}) > 0$ (1 point). Therefore, the minimum of the $l(\theta)$ happens at $\theta = \tan^{-1}\frac{1}{2}$ and we can compute the answer $l(\tan^{-1}\frac{1}{2}) = 10\sqrt{5}$.(4 points)

6. (17%) Let $h(x) = x^{1/3}(x-4)$. Then $h'(x) = \frac{4(x-1)}{3x^{2/3}}$ and $h''(x) = \frac{4(x+2)}{9x^{5/3}}$. Answer the following questions by filling each blank below. Show your work (computations and reasoning) in the space following. Put **None** in the blank if the item asked does **not** exist, each blank is worth 2 pts.

(a) The function is increasing on the interval(s) ______________________ and decreasing on the interval(s) ______________________

The local maximal point(s) (x, y) = ______________________ and

The local minimal point(s) (x, y) = ______________________.

(b) The function is concave upward on the interval(s) ______________________ and concave downward on the interval(s) ______________________. The inflection point(s) (x, y) = ______________________.

(c) Sketch the graph of the function. Indicate, if any, where it is increasing/decreasing, where it concaves upward/downward, all relative maxima/minima, inflection points and asymptotic line(s) (if any). (3%)

Solution:

(a)

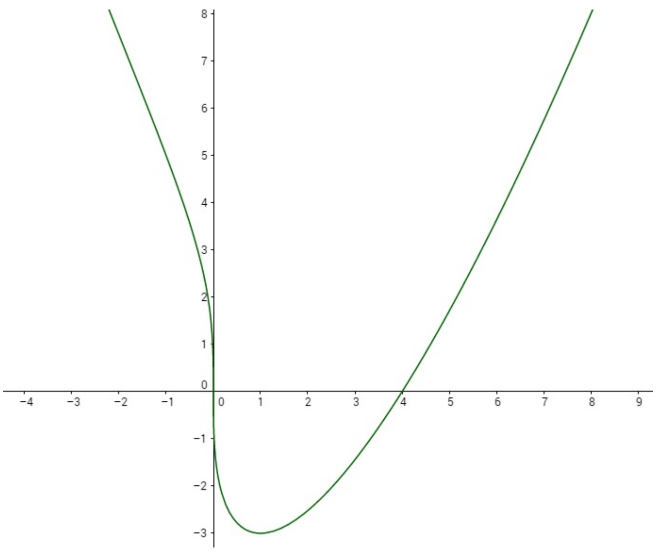
1. (2pt) $(1, \infty)$
2. (2pt) $(-\infty, 1)$
3. (2pt) None
4. (2pt) $(1, -3)$

(b)

1. (2pt) $(-\infty, -2), (0, \infty)$ / $(-\infty, -2) \cup (0, \infty)$
2. (2pt) $(-2, 0)$
3. (2pt) $(-2, 6\sqrt[3]{2}), (0, 0)$

(c)

1. (1pt) Mark all 4 points to get this point: $(-2, 6\sqrt[3]{2}), (0, 0), (-1, 3), (0, 4)$
2. (1pt) Draw monotonicity and concavity correct and do not draw any asymptote to get this point.
3. (1pt) Draw something like a curve to get this point.



評分標準:

- You do not lose any points if you replace any open end by closed end, e.g. $[-2,0]$. But if you interchange the two ends, e.g. $(0,-2)$, you lose 1pt for each blank.
- Misplacing (a) 1. $\leftrightarrow$ 2. (a) 3. $\leftrightarrow$ 4. (b) 1. $\leftrightarrow$ 2. costs 2pt each pair.
- In (a) 2. $(-\infty,0),(0,1)$ is not correct, but won't lose points.
- If (a) 4. correct and (a) 3. empty, you get 1pt for (a) 3.
- In (b) 3. missing (0, 0) costs 1pt.
- If you think the domain of $x^{1/3}$ is $[0,\infty)$, use the following grading:
  (a) 1. (2pt) $(1,\infty)$ 2. (1pt) $(0,1)$ 3. (1pt) $(0,0)$ or None 4. (2pt) $(1,-3)$
  (b) 1. (1pt) $(0,\infty)$ 2. (1pt) None 3. (1pt) None (c) (3pt) right half graph.
  You get at most 12pt in this case.
- In (c), if you write $x=0$ a vertical asymptote (it is actually a vertical tangent line), you lose the point of 2.
- Note that $7<6\sqrt[3]{2}<8$. Since we have grids for graphing, draw the point $(-2,6\sqrt[3]{2})$ between 7 and 8 or you lose the point of 1.

Remarks

- $-\infty$ and ∞ is not a real number. We don't use closed end like $[1,\infty]$ in real number system.
- Use $(-\infty,-2)\cup(0,\infty)$. $(-\infty,-2)\cap(0,\infty)=\varnothing$
- A function f is (strictly) decreasing on (a,b) if:
  For any $x_1,x_2\in(a,b)$, $x_1<x_2 \implies f(x_1)>f(x_2)$.
  We use first derivative just for test if f is differentiable. When f is not differentiable, you should check the original definition. Since $(0,0)$ exists, the interval $(-\infty,1)$ has this property. Basically we write the largest interval as solution. Thanks to professor for not losing points.
- Similarly, since $(0,0)$ exists, it is a inflection point, although $h''(0)$ does not exist.
- In real calculus, we define $x^{1/3}$ to be the inverse function of x^3. Then the domain of $x^{1/3}$ is the whole real number line, while x^b is generally well defined only on $x>0$ given any real number b. (But x^1 is good on $(-\infty,\infty)$, right?)
- $x=0$ is a vertical tangent line at $(0,0)$, so the curve should tangent to it. This do not cost any points since in Textbook we do not mention this.

7. Let the curve $x^2y^2 + 2xy = 8$ be given.
   (a) (4%) Express y' in terms of x and y.
   (b) (4%) Find points on the curve with $y = 2$ and the tangent lines at these points.
   (c) (4%) Find y'' at the points in (b).

Solution:

(a) (4 points) Do implicit derivative on variable x,

$$\begin{cases} (2xy^2 + 2x^2yy') + (2y + 2xy') = 0 \\ (x^2y + x)y' = -y - xy^2 \end{cases}$$

(3 points), each mistakes will minus 1 point

$$y' = -\frac{xy^2 + y}{x^2y + x} = -\frac{y(xy+1)}{x(xy+1)} = -\frac{y}{x} \quad \text{(1 point)}$$

(b) (4 points) Find points on the curve with $y = 2$
$x^2 \cdot 4 + 4x = 8$, so we have $x^2 + x - 2 = 0 = (x+2)(x-1)$
Hence the intersection points are

$$P_1 = (-2, 2)\text{(1 point)}, \quad P_2 = (1, 2)\text{(1 point)}$$

$$m_1 = -\frac{(-2)\cdot 4 + 2}{(-2)^2 \cdot 2 + (-2)} = \frac{6}{6} = 1 \quad \text{(1 point)}$$

$$m_2 = -\frac{4+2}{2+1} = -2 \quad \text{(1 point)}$$

Tangent line at P_1: $y - 2 = x + 2$
Tangent line at P_2: $y - 2 = -2(x-1)$

(c) (4 points) From (1), we have

$$y'' = -\frac{y'x - y}{x^2} = \frac{2y}{x^2}$$

At P_1, $y'' = \frac{2\cdot 2}{(-2)^2} = 1$ (2 points)
At P_2, $y'' = \frac{4}{1} = 4$ (2 points)

8. Find the derivative of the following functions.

(a) (4%) $y = \left(\tan^{-1} x\right)^{\sin x}$, $x > 0$.

(b) (4%) $y = \log_{e^x}(\tan x)$, $0 < x < \frac{\pi}{2}$.

(c) (4%) $y = \frac{(2x+1)^5(x^2+1)^3}{(3x-2)^6(x^3+1)^4}$, find $y'(0)$.

Solution:

(a)

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sin x \cdot \ln(\tan^{-1} x)} \qquad [1 \text{ points}]$$

$$= (\tan^{-1} x)^{\sin x}(\cos x \cdot \ln(\tan^{-1} x) + \frac{\sin x}{(1+x^2)\cdot \tan^{-1} x}) \qquad [3 \text{ points}]$$

(b)

$$y = \frac{\ln \tan x}{\ln e^x} = \frac{\ln \tan x}{x} \qquad [1 \text{ points}]$$

hence by quotient rule,

$$\frac{dy}{dx} = \frac{\frac{x\sec^2 x}{\tan x} - \ln \tan x}{x^2} \qquad [3 \text{ points}]$$

(c)

We can write y as

$$y = (2x+1)^5(x^2+1)^3(3x-2)^{-6}(x^3+1)^{-4}$$

hence by product rule,

$$\begin{aligned} y' = & 10(2x+1)^4(x^2+1)^3(3x-2)^{-6}(x^3+1)^{-4}) \\ & +6x(2x+1)^5(x^2+1)^2(3x-2)^{-6}(x^3+1)^{-4} \\ & -18(2x+1)^5(x^2+1)^3(3x-2)^{-7}(x^3+1)^{-4} \\ & -12x^2(2x+1)^5(x^2+1)^3(3x-2)^{-6}(x^3+1)^{-5} \end{aligned}$$

[3 points]

$$y'(0) = 10\cdot\frac{1}{64} - 18\cdot(-\frac{1}{128}) = \frac{19}{64} \qquad [1 \text{ points}]$$

(The other way)

$$\ln y = 5\ln(2x+1) + 3\ln(x^2+1) - 6\ln(3x-2) + 4\ln(x^3+1)$$

hence,

$$\frac{y'}{y} = \frac{10}{2x+1} + \frac{6x}{x^2+1} - \frac{18}{3x-2} + \frac{12x^2}{x^3+1} \qquad [3 \text{ points}]$$

$$y'(0) = y(0)(10+0+9+0) = \frac{19}{64} \qquad [1 \text{ points}]$$

[Grading Criterion]

(a)(b)

Simply the functions. [1 points]

Differentiate the equation correctly. [3 points]

(c)

Differentiate the equation correctly. [3 points]

Get the correct value of $f'(0)$.[1 points]