1051微甲06-10班期末考解答和評分標準

1. (13%) Find the orthogonal trajectories of the family of curves $y = \tan^{-1}(kx)$, where k is an arbitrary constant.

Solution:

For the original curves:

$\frac{dy}{dx} = \frac{k}{1+k^2x^2} = \frac{\frac{\tan y}{x}}{1+\tan^2 y} = \frac{\sin y \cos y}{x}$

The curves that we wanted:

$\frac{dy}{dx} = \frac{-x}{\sin y \cos y}$

$-xdx = \sin y \cos y dy \Longrightarrow \int(-x)dx = \int \sin y \cos y dy \Longrightarrow x^2 = \cos^2 y + C$,

where C is an arbitrary constant.

Differentiating the original curves to get the slope: 3 points.

Cancelling the constant k: 2 points.

Writing down the slope of the curves we wanted: 2 points.

Solving the differential equation: 6 points.

2. (12%) Solve the initial value problem

$$\begin{cases} x^2y' - y = 2xe^{-\frac{1}{x}} \ln x, \ x > 0 \\ y(1) = 2 \end{cases}$$

Solution:

Multiplying $\frac{1}{x^2}$ both sides of the equation, we have

$$y' - \frac{1}{x^2}y = \frac{2}{x}e^{-\frac{1}{x}} \ln x$$

which is a linear equation and the integrating factor $I(x)$ is

$$I(x) = \exp\left(\int -\frac{1}{x^2}\,dx\right) = e^{\frac{1}{x}}. \qquad (4\%)$$

Hence

$$\begin{aligned} y &= \frac{1}{I(x)} \int I(x) \cdot \frac{2}{x} e^{-\frac{1}{x}} \ln x\,dx \\ &= e^{-\frac{1}{x}} \int \frac{2}{x} \ln x\,dx \qquad (\text{Let } u = \ln x, du = \frac{1}{x}\,dx) \\ &= e^{-\frac{1}{x}} \int 2u\,du \\ &= e^{-\frac{1}{x}} (u^2 + c) \\ &= e^{-\frac{1}{x}} \left[(\ln x)^2 + c\right] \qquad (4\%) \end{aligned}$$

By initial condition,

$$y(1) = e^{-1}c = 2 \qquad \Rightarrow \qquad c = 2e \qquad (4\%)$$

Therefore the solution is

$$\boxed{y = e^{-\frac{1}{x}}\left[(\ln x)^2 + 2e\right]}$$

$(e^{\frac{1}{x}}y = \sim : -1\%)$

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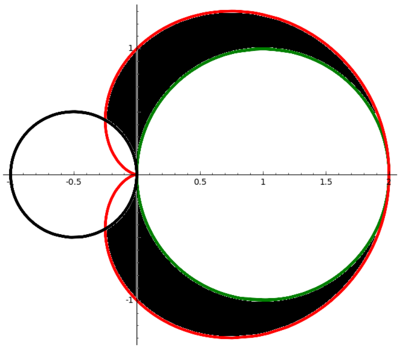
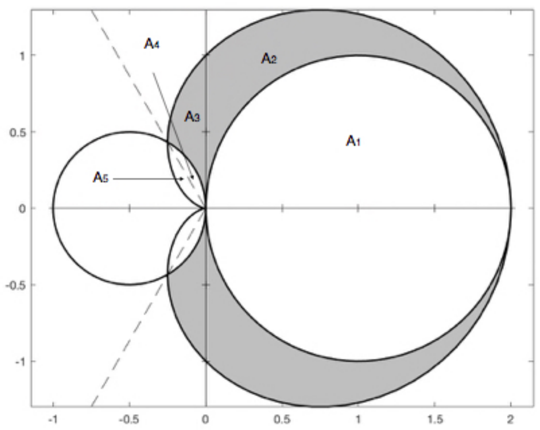
3. (13%) Find the area of the region that lies inside the curve $r = 1 + \cos\theta$ but outside the curves $r = 2\cos\theta$ and $r = -\cos\theta$.



Solution:



By symmetry, we only need to compute the area $A_2 + A_3$, then the answer will be $2(A_2 + A_3)$.
The intersection points of $r = 1 + \cos\theta$ and $r = -\cos\theta$

$$1 + \cos\theta = -\cos\theta \Rightarrow 2\cos\theta = -1 \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ with } r = \frac{1}{2} \ \textbf{(2\%)}$$

Note that the dash line is $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ and A_4 is different from A_5
solution 1

(1) Compute A_2 first.
(method 1)

$$A_2 = (A_2 + A_1) - A_1 = \underbrace{\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos\theta)^2 d\theta}_{A_2+A_1} - \underbrace{\int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 d\theta}_{A_1} \ \textbf{(2\%)}$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos\theta)^2 - (2\cos\theta)^2 d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + 2\cos\theta - 3\cos^2\theta d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + 2\cos\theta - \frac{3(1+\cos(2\theta))}{2} d\theta$$
$$= \frac{1}{2}(-\frac{\theta}{2} + 2\sin\theta - \frac{3\sin(2\theta)}{4})\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}(-\frac{\pi}{4} + 2) = 1 - \frac{\pi}{8} \ \textbf{(3\%)}$$

(method 2)
$A_1 = \frac{\pi}{2}$ because it is half of the area of a circle with radius=1 **(2%)**

$$\begin{aligned}A_2 + A_1 &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin\theta + \frac{\sin(2\theta)}{4}\right)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{2}\left(\frac{3\pi}{4} + 2\right) \ \textbf{(2\%)} \\ A_2 &= \frac{1}{2}\left(\frac{3\pi}{4} + 2\right) - A_1 = 1 - \frac{\pi}{8} \textbf{(1\%)}\end{aligned}$$

(2) Compute A_3
(method 1)

$$\begin{aligned}A_3 &= (A_3 + A_4) - A_4 = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(1+\cos\theta)^2 \mathrm{d}\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(-\cos\theta)^2 \mathrm{d}\theta \ \textbf{(2\%)} \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 1 + 2\cos\theta \mathrm{d}\theta \\ &= \frac{1}{2}\,(\theta + 2\sin\theta)\Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \\ &= \frac{1}{2}\left(\frac{\pi}{6} + \sqrt{3} - 2\right) \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \ \textbf{(2\%)}\end{aligned}$$

(method 2)

$$\begin{aligned}A_3 &= (A_3 + A_4 + A_5) - A_4 - A_5 \\ &= \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos\theta)^2 \mathrm{d}\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-\cos\theta)^2 \mathrm{d}\theta - \int_{\frac{2\pi}{3}}^{\pi} (1+\cos\theta)^2 \mathrm{d}\theta \ \textbf{(2\%)} \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2}\right)\mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2} + \frac{\cos(2\theta)}{2}\mathrm{d}\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2}\right)\mathrm{d}\theta \\ &= \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin\theta + \frac{\sin(2\theta)}{4}\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} - \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin\theta + \frac{\sin(2\theta)}{4}\right)\Big|_{\frac{2\pi}{3}}^{\pi} \\ &= \frac{1}{2}\left(\frac{3\pi}{4} - 2\right) - \frac{1}{2}\left(\frac{\pi}{12} + \left(\frac{-\sqrt{3}}{8}\right)\right) - \frac{1}{2}\left(\frac{\pi}{2} + (-\sqrt{3}) + \left(-\frac{-\sqrt{3}}{8}\right)\right) \\ &= \underbrace{\left(\frac{3\pi}{8} - 1\right)}_{A_3+A_4+A_5} - \underbrace{\left(\frac{\pi}{24} - \frac{\sqrt{3}}{16}\right)}_{A_4} - \underbrace{\left(\frac{\pi}{4} - \frac{7\sqrt{3}}{16}\right)}_{A_5} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \ \textbf{(2\%)}\end{aligned}$$

(3) Answer $= 2 \times (A_2 + A_3) = 2 \times (1 - \frac{\pi}{8} + \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1) = \sqrt{3} - \frac{\pi}{12}$ **(2%)** □

solution 2

$$\begin{aligned}A_2 + A_3 &= (A_1 + A_2 + A_3 + A_4) - A_1 - A_4 \\ &= \int_0^{\frac{2\pi}{3}} \frac{1}{2}(1+\cos\theta)^2 \mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(-\cos\theta)^2 \mathrm{d}\theta \ \textbf{(3\%)} \\ &= \left(\frac{3\theta}{4} + \sin\theta + \frac{\sin(2\theta)}{8}\right)\Big|_0^{\frac{2\pi}{3}} - \left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}} - \left(\frac{\theta}{4} + \frac{\sin(2\theta)}{8}\right)\Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \\ &= \left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{16}\right)\right) - \left(\frac{\pi}{2}\right) - \left(\frac{\pi}{24} - \frac{\sqrt{3}}{16}\right) \\ &= \underbrace{\left(\frac{\pi}{2} + \frac{7\sqrt{3}}{16}\right)}_{A_1+A_2+A_3+A_4} - \underbrace{\left(\frac{\pi}{2}\right)}_{A_1} - \underbrace{\left(\frac{\pi}{24} - \frac{\sqrt{3}}{16}\right)}_{A_4} \ \textbf{(6\%)} \\ &= \frac{\sqrt{3}}{2} - \frac{\pi}{24}\end{aligned}$$

Answer $= 2 \times (A_2 + A_3) = \sqrt{3} - \dfrac{\pi}{12}$ **(2%)** □

4. (10%) Find the arc length of the curve. $x = \cos t + \ln(\tan \frac{1}{2}t)$, $y = \sin t$, $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

Solution:

$$x = \cos t + \ln(\tan \frac{1}{2}t) \qquad \frac{dx}{dt} = -\sin t + \frac{\sec^2 \frac{1}{2}t}{2\tan\frac{1}{2}t} = -\sin t + \frac{1}{\sin t}$$

$$y = \sin t \qquad \frac{dy}{dt} = \cos t$$

$$\begin{aligned}
\therefore \text{Arc length } &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\csc^2 t - 1}\, dt \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left|\cot t\right| dt \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\, dt - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cot t\, dt \\
&= \ln|\sin t|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \ln|\sin t|\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \\
&= \ln 2
\end{aligned}$$

5. (20%) Let R be the region bounded by the x-axis, $x = e$ and the curve $y = \ln x$.

(a) (5%) Find the volume of the solid obtained by revolving R about the x-axis.

(b) (5%) Find the volume of the solid obtained by revolving R about the y-axis.

(c) (5%) Find the centroid of R.

(d) (5%) Find the volume of the solid obtained by revolving R about $x + y = 1$.

Solution:

(a) $= \int_1^e \pi(\ln x)^2\ dx(2pts) = \pi[x(\ln x)^2|_1^e - \int_1^e 2\ln x\ dx] = \pi(e-2)$
$(\int_1^e \ln x\ dx = x\ln x - x|_1^e = e - e + 1 = 1)$

(b) $= \int_1^e 2\pi x \ln x\ dx(2pts) = 2\pi[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2]|1^e = \frac{\pi}{2}(e^2+1)$

(c) $A = \int_1^e lnx\ dx = 1$ (1 pts)
$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx = \int_1^e x\ln x\ dx = \frac{e^2+1}{4}$ (2 pts)
$\bar{y} = \int_0^1 y(e-e^y)\ dy = \int_0^1 ey - ye^y\ dy = \frac{e-2}{2}$ (2 pts)
Thus $(\bar{x},\bar{y}) = (\frac{e^2+1}{4}, \frac{e-2}{2})$

(d) By Pappus's centroid theorem $V = A\cdot 2\pi d$ (3 pts)
Calculate d: $(\frac{e^2+1}{4}+t, \frac{e-2}{2}+t)$ is on $x+y=1$
$\Rightarrow t = \frac{7-e^2-2e}{8} \Rightarrow d = \frac{e^2+2e-7}{4\sqrt{2}}$
Then $V = \pi\frac{e^2+2e-7}{2\sqrt{2}}$

6. (10%) Compute the area of the surface generated by rotating the curve $y = \ln x$, $0 \le x \le 1$ about the y-axis.

Solution:

The surface area S is:

$$S = \int_{-\infty}^{0} 2\pi e^{y}\sqrt{1+e^{2y}}\,\mathrm{dy} = \int_{0}^{1} 2\pi x\sqrt{1+\frac{1}{x^2}}\,\mathrm{dx} \qquad \text{(6 pts)}$$

$$= 2\pi \int_{0}^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\,\sec^2\theta\,\mathrm{d}\theta \qquad \text{(1 pt for change variable: } x = \tan\theta\text{)}$$

$$= 2\pi \int_{0}^{\frac{\pi}{4}} \sec^3\theta \mathrm{d}\theta = \frac{2\pi}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\Big|_0^{\pi/4} \qquad \text{(2 pts)}$$

$$= \pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \qquad \text{(1 pt)}$$

7. (12%) Evaluate the following integrals.

(a) (5%) $\int_0^3 \frac{x^2}{\sqrt{x+1}} dx.$

(b) (7%) $\int \frac{3x^3 - 2x - 2}{x^2(x^2+1)} dx.$

Solution:

(a) Let $u = x + 1$. Then $dx = du$. Thus the Substitution Rule gives

$$\begin{aligned}\int_0^3 \frac{x^2}{\sqrt{x+1}} &= \int_1^4 \frac{(u-1)^2}{\sqrt{u}} du \\ &= \int_1^4 u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}} du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \Big|_{u=1}^{u=4} \\ &= \frac{76}{15}.\end{aligned}$$

(b) The partial fraction decomposition of the rational function is

$$\frac{3x^3 - 2x - 2}{x^2(x^2+1)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{5x+2}{x^2+1}.$$

Thus,

$$\begin{aligned}\int \frac{3x^3 - 2x - 2}{x^2(x^2+1)} dx &= \int \frac{-2}{x} + \frac{-2}{x^2} + \frac{5x+2}{x^2+1} dx \\ &= -2\ln|x| + 2x^{-1} + \frac{5}{2}\ln(x^2+1) + 2\tan^{-1} x + C.\end{aligned}$$

[Grading Criterion]

(a) correct change of variable :1 point, antiderivative :3 points, answer :1 point.

(b) partial fraction :1 point, the four terms in the answer :1/1/2/2 points respectively.

8. (10%) Evaluate the following improper integrals.

(a) (5%) $\int_e^\infty \frac{1}{x(\ln x)^2} dx.$

(b) (5%) $\int_0^1 \frac{1}{x+\sqrt{x}} dx.$

Solution:

(a) (total 5 points)

$$\int_e^\infty \frac{1}{x(ln(x))^2} dx$$

$$= \lim_{t\to\infty} \int_e^t \frac{1}{x(ln(x))^2} dx \qquad \text{(2 points)}$$

Let $u = e^x$, then $dx = \frac{1}{u} du$, then

$$= \lim_{t\to\infty} \int_1^{ln(t)} \frac{1}{u^2} du \qquad \text{(1 point)}$$

$$= \lim_{t\to\infty} [\frac{-1}{u}]_1^{ln(t)}$$

$$= \lim_{t\to\infty} (1 - \frac{1}{ln(t)})$$

$$= 1 \qquad \text{(2points)}$$

(b) (total 5 points)

$$\int_0^1 \frac{1}{x+\sqrt{x}} dx$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x+\sqrt{x}} dx \qquad \text{(2 points)}$$

since $\frac{1}{x+\sqrt{x}} \to \infty$ as $x \to 0^+$

Let $u = \sqrt{x}$, $dx = 2udu$

$$= \lim_{t\to 0^+} \int_{\sqrt{t}}^1 \frac{2u}{u^2+u} du \qquad \text{(1 point)}$$

$$= \lim_{t\to 0^+} \int_{\sqrt{t}}^1 \frac{2}{u+1} du$$

$$= \lim_{t\to 0^+} [2ln(u+1)]_{\sqrt{t}}^1$$

$$= 2ln(2) - \lim_{t\to 0^+} 2ln(\sqrt{t}+1)$$

$$= 2ln(2) \qquad \text{(2 points)}$$