1. (10%) Find the following limits if they exist.

(a) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+|t|}}-\frac{1}{t}\right)$

(b) $\lim_{x\to\infty}\sqrt{2x^2+1}\sin\left(\frac{1}{x}\right)$

Solution:

(a) First we compute it by rationalization.

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+|t|}}-\frac{1}{t}\right)=\lim_{t\to 0}\left(\frac{1-\sqrt{1+|t|}}{t\sqrt{1+|t|}}\right)=\lim_{t\to 0}\left(\frac{1-\sqrt{1+|t|}}{t\sqrt{1+|t|}}\times\frac{1+\sqrt{1+|t|}}{1+\sqrt{1+|t|}}\right)$$
$$=\lim_{t\to 0}\left(\frac{-|t|}{t\sqrt{1+|t|}\big(1+\sqrt{1+|t|}\big)}\right).\ \textbf{(2 points)}$$

For a further calculation, we separate it in two cases: $t\to 0^+$ **(1 point)** and $t\to 0^-$ **(1 point)**. So we obtain

$$\lim_{t\to 0^+}\left(\frac{-|t|}{t\sqrt{1+|t|}\big(1+\sqrt{1+|t|}\big)}\right)=\lim_{t\to 0^+}\left(\frac{-1}{\sqrt{1+t}\big(1+\sqrt{1+t}\big)}\right)=-1/2.$$

(1 point)

$$\lim_{t\to 0^-}\left(\frac{-|t|}{t\sqrt{1+|t|}\big(1+\sqrt{1+|t|}\big)}\right)=\lim_{t\to 0^+}\left(\frac{1}{\sqrt{1-t}\big(1+\sqrt{1-t}\big)}\right)=1/2.$$

(1 point)

Thus the limit does not exists.

You may separate it in the two cases before doing rationalization, and you will get 1 point respectively in each correct computation of rationalization.

(b)

$$\lim_{x\to\infty}\left(\sqrt{2x^2+1}\sin(\frac{1}{x})\right)=\lim_{x\to\infty}\left(\frac{\sqrt{2x^2+1}}{x}\times\frac{\sin(1/x)}{1/x}\right)\ \textbf{(2 points)}$$
$$=\lim_{x\to\infty}\left(\sqrt{2+\frac{1}{x^2}}\right)\times 1=\sqrt{2}\ \textbf{(2 points)}.$$

You may substitute $x\to\infty$ into $t\to 0^+$. Here you will lose 1 point for a wrong substitution.

2. (10%) Find the following limits if they exist.

(a) $\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}}$

(b) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

Solution:

(a) 1. Let $y(x)=(1-\cos x)^{\frac{1}{\ln x}}$, then $\ln y(x)=\ln(1-\cos x)^{\frac{1}{\ln x}}=\frac{\ln(1-\cos x)}{\ln x}$ We use the L'Hospital Rule to get

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x} &\overset{(\frac{\infty}{\infty}),L'}{=}\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{x^2}{1-\cos x}(2\%)\\ &=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\frac{x^2}{1-\cos x}\\ &=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\frac{4(\frac{x}{2})^2}{2\sin^2(\frac{x}{2})}\\ &=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot 2(\lim_{x\to 0^+}\frac{\frac{x}{2}}{\sin\frac{x}{2}})^2(3\%)\\ &=1\cdot(2\cdot 1^2)\\ &=2(4\%)\end{aligned}$$

$\lim_{x\to 0^+}\ln y(x)=\ln(\lim_{x\to 0^+}y(x))=2$

Thus, $\lim_{x\to 0^+}y(x)=\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}}=e^2(5\%)$

2.

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x} &\overset{(\frac{\infty}{\infty}),L'}{=}\lim_{x\to 0^+}\frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}(2\%)=\lim_{x\to 0^+}\frac{x\sin x}{1-\cos x}\\ &\overset{(\frac{\infty}{\infty}),L'}{=}\lim_{x\to 0^+}\frac{\sin x+x\cos x}{\sin x}(3\%)=\lim_{x\to 0^+}(1+\frac{x\cos x}{\sin x})=1+\lim_{x\to 0^+}\frac{x\cos x}{\sin x}\\ &\overset{(\frac{0}{0}),L'}{=}1+\lim_{x\to 0^+}\frac{\cos x-x\sin x}{\cos x}=1+\lim_{x\to 0^+}(1-\frac{x\sin x}{\cos x})\\ &=1+1-0=2(4\%)\end{aligned}$$

So the answer is $e^2(5\%)$

3.

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x} &=\lim_{x\to 0^+}\frac{\ln 2\sin^2\frac{x}{2}}{\ln x}\overset{(\frac{\infty}{\infty}),L'}{=}\lim_{x\to 0^+}\frac{\frac{1}{2\sin^2\frac{x}{2}}\cdot\sin\frac{x}{2}\cdot\frac{1}{2}}{\frac{1}{x}}(2\%)\\ &=\lim_{x\to 0^+}\frac{x}{\sin\frac{x}{2}}=2\lim_{x\to 0^+}\frac{\frac{x}{2}}{\sin\frac{x}{2}}=2\cdot 1=2(4\%)\\ \lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}} &=e^{\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x}}\because e^t\text{ is continuous}\\ &=e^2(5\%)\end{aligned}$$

(b) 1. Use L'Hospital Rule

$$\begin{aligned}\lim_{x\to 1}(\frac{1}{\ln x}-\frac{1}{x-1}) &=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}\overset{(\frac{0}{0}),L'}{=}\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+(x-1)\cdot\frac{1}{x}}(2\%)\\ &=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}\overset{(\frac{0}{0}),L'}{=}\lim_{x\to 1}\frac{1}{\ln x+x\cdot\frac{1}{x}+1}(4\%)=\frac{1}{2}(5\%)\end{aligned}$$

2. Let $y=x-1$, then $y\to 0$ as $x\to 1$, so

$$\begin{aligned}\lim_{x\to 1}(\frac{1}{\ln x}-\frac{1}{x-1}) &=\lim_{y\to 0}(\frac{1}{\ln(1+y)}-\frac{1}{y})=\lim_{y\to 0}\frac{y-\ln(1+y)}{y\ln(1+y)}\\ &\overset{(\frac{0}{0}),L'}{=}\lim_{y\to 0}\frac{1-\frac{1}{1+y}}{\ln(1+y)+y\frac{1}{1+y}}(2\%)=\lim_{y\to 0}\frac{y}{(1+y)\ln(1+y)+y}\\ &=\lim_{y\to 0}\frac{1}{(1+y)\frac{\ln(1+y)}{y}+1}(3\%)=\frac{1}{(1+0)\cdot 1+1}=\frac{1}{2}(4\%)\end{aligned}$$

Remark.
$$\lim_{y\to 0}\frac{\ln(1+y)}{y}=\lim_{y\to 0}\frac{\ln(1+y)-\ln(1+0)}{y-0}=\frac{d}{dy}\ln(1+y)\Big|_{y=0}=\frac{1}{1+y}\Big|_{y=0}=1(5\%)$$

3.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}\overset{(\frac{0}{0}),L'}{=}\lim_{x\to 1}\frac{1-\frac{1}{x}}{(x-1)\frac{1}{x}+\ln x}(2\%)$$
$$\overset{(\frac{0}{0}),L'}{=}\lim_{x\to 1}\frac{\frac{1}{x^2}}{(x-1)\frac{-1}{x^2}+\frac{1}{x}+\frac{1}{x}}(4\%)$$
$$=\frac{\frac{1}{1}}{(1-1)\frac{-1}{1^2}+\frac{1}{1}+\frac{1}{1}}=\frac{1}{2}(5\%)$$

3. (12%) Student A used some mathematical software to plot a Lapras-like curve (乘龍) as Figure 1.

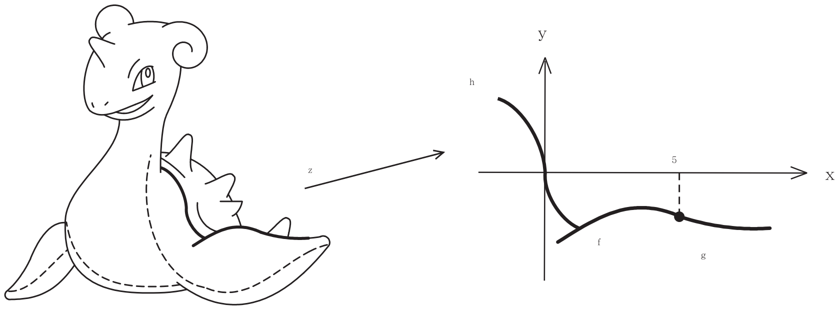

Figure 1: Lapras-like curve.

(a) On Lapras' body, he considered two functions:

$$f(x) = -\frac{3}{10}(x-4)^2 - \frac{17}{10}, \qquad 0.3 < x \le 5$$

$$g(x) = \frac{a}{x-4} + bx - \frac{22}{5}, \qquad 5 < x < 8.$$

Find constants a and b such that the union of $f(x)$ and $g(x)$ is differentiable at $x = 5$.

(b) The upper part of Lapras' back is depicted by $h(x)$. Which of the following three functions can be a good candidate for $h(x)$? Give reasons why other two functions are not good candidates.

$$\frac{3}{2}\left(\cos^{-1} x - \frac{\pi}{2}\right), \qquad -2x^{\frac{1}{3}}, \qquad \sinh x.$$

Solution:

(a)

1. (1pt) $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} g(x) = f(5)$
2. (1pt) $-\frac{3}{10}(5-4)^2 - \frac{17}{10} = \frac{a}{5-4} + 5b - \frac{22}{5} \implies a + 5b = \frac{12}{5}$
3. (2pt) $\lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^+} \frac{g(x)-f(5)}{x-5}$
4. (2pt) Either (i) use *L'Hôpital's Rule* or (ii) compute directly

   (i) $\implies \lim_{x\to 5^-} \frac{f'(x)}{1} = \lim_{x\to 5^+} \frac{g'(x)}{1} \implies -\frac{6}{10}(5-4) = \frac{-a}{(5-4)^2} + b$

   (ii) LHS $= \lim_{x\to 5^-} \frac{-\frac{3}{10}(x-4)^2 - \frac{17}{10} - (-2)}{x-5} = \lim_{x\to 5^-} -\frac{3}{10}\frac{(x-3)(x-5)}{x-5}$

   RHS $= \lim_{x\to 5^+} \frac{\frac{a}{x-4} + bx - \frac{22}{5} - (-2)}{x-5} = \lim_{x\to 5^+} \frac{(\frac{a}{x-4} - a) + (bx - 5b)}{x-5} = \lim_{x\to 5^+} \frac{\frac{a(5-x)}{x-4} + b(x-5)}{x-5}$

   $\implies -a + b = -\frac{3}{5}$
5. (2pt) $a + 5b = \frac{12}{5}$, $-a + b = -\frac{3}{5} \implies a = \frac{9}{10}$, $b = \frac{3}{10}$

(b)

- (1pt) $s(x) = \sinh(x)$ is not a good candidate.
  (1pt) Any of following reason get the point.
  (i) Draw a correct graph of $s(x)$
  (ii) $s(x) > 0$ when $x > 0$, while $h(x) < 0$ in this case.
  (iii) $s'(0) = \cosh(0) = 1$, while $h'(0)$ is (nearly) negative infinite.
  (iv) $s'(x) = \cosh(x) \ge 1$, so $s(x)$ is increasing, while $h(x)$ is decreasing.
- (1pt) $c(x) = \frac{3}{2}\left(\cos^{-1}(x) - \frac{\pi}{2}\right)$ is not a good candidate.
  (1pt) Any of following reason get the point.
  (i) Draw a correct graph of $c(x)$
  (ii) $c'(0) = \frac{3}{2}\frac{-1}{\sqrt{1-x^2}}\Big|_0 = -\frac{3}{2}$, while $h'(0)$ is (nearly) negative infinite.
  (iii) $c''(x) = \frac{3}{4}\frac{-2x}{(1-x^2)^{3/2}} < 0$ when $x > 0$, so $c(x)$ is concave downward on $(0, 1)$, while $h(x)$ is concave

upward on $(0,1)$.

(iv) The global minimum of $c(x)$ is $c(1) = -\frac{3}{4}\pi \approx -2.36$, while $f(1) = -4.4$ and even smaller when $x < 1$. Thus $c(x)$ do not connect to $f(x)$.

評分標準:

(a)

- 1. and 2. are continuous part; 3. and 4. are differentiable part.
- Unless you extended the function or explicitly defined them, any of $g(5), f'(5), g'(5), f(x) = g(x), f'(x) = g'(x)$ costs 1pt due to their domain.
- it's OK to omit 1. and 2. since it is recovered when calculating 3. (but then you cannot use *L'Hôpital's Rule* before claiming 0/0). However, if you use $g(5)$ instead of $f(5)$ and substitute $g(x)$ with 5 (rather then $g(5) = f(5) = -2$), or just use $f'(5) = g'(5)$, you would miss the continuous part.
- If you write (**) $\implies$ (***) (or any words like "If", "then", "implies"), you lose 1pt for false implication. If no explicit implication is written, you won't lose any points if you replace (**) by (***). (See Remarks below)
- If you have a stupid mistake like $-2 + \frac{22}{5} = \frac{2}{5}$, you get 2pt of 5. even the answer is incorrect. However, having mistakes related to calculus like $(\frac{a}{x-4})' = \frac{a}{(x-4)^2}$ would lost 2pt of 5..
- In case of no correct points, you get 1pt if you write something.
  Draw a pikachu to get 3pt. (No one did that, though. Sad.)

(b)

- Having any mistake in reason part would lose the point.
- Writing $-2x^{1/3}$ not a good candidate get no points including reason part.
- A too big domain is not a problem. We can restrict it to suitable domain.
- $\sinh(x) \to \infty$ when $x \to \infty$ is not a good reason. $x^3 - x$ also did that but the slope of tangent line at 0 is negative in this case.

Remarks

- The domain of $f(x)$ is $(0.3, 5]$; $g(x)$ is $(5, 8)$; $f'(x)$ is $(0.3, 5)$; $g'(x)$ is $(5, 8)$.
- $g(x)$ is not a polynomial. It is a rational function, i.e., quotient of two polynomials. It is C^∞ (∞-differentiable) except where denominator polynomial is zero.
- (*) $F(x)$ is continuous at $x = 5$
  (**) $\lim_{x\to 5^-} \frac{F(x) - F(5)}{x-5} = \lim_{x\to 5^+} \frac{F(x) - F(5)}{x-5}$ or $F'_-(5) = F'_+(5)$
  (***) $\lim_{x\to 5^-} F(x) = \lim_{x\to 5^+} F(x)$ or $F'(5^-) = F'(5^+)$
  (****) Both limits in (***) exists.
  Note that (**) means $F(x)$ is differentiable at 5, while (***) means limit of $F'(x)$ at 5 exists. We have (*)+(***)+(****) $\implies$ (**), a sufficient but not necessary condition since (****) may not holds. This can be proved by *L'Hôpital's Rule*.
  A counter-example of (**) $\implies$ (***) is $G(x) = x^2 \sin(\frac{1}{x}), G(0) = 0$.
- In real calculus, we define $x^{1/3}$ to be the inverse function of x^3. Then the domain of $x^{1/3}$ is the whole real number line, while x^b is generally well defined only on $x > 0$ given any real number b. (But x^1 is good on $(-\infty, \infty)$, right?)

4. (10%)

(a) Find $\frac{dy}{dx}$ at $(x, y) = (\pi, 0)$, where $\tan(x - y) = \frac{y}{1+x^2}$.

(b) Find $\frac{dy}{dx}$ where $x^{y^2} = y^{x^2}$, $x > 0$, $y > 0$.

Solution:

(a) First we differentiate the both sides of the equation, then we have

$$\sec^2(x-y)(1-y') \ \textbf{(1 point)} = \frac{y'}{1+x^2} + \frac{-2xy}{(1+x^2)^2} \ \textbf{(1 point)}.$$

Now take $(x, y) = (\pi, 0)$, we obtain

$$\sec^2(\pi)(1-y') = \frac{y'}{1+\pi^2},$$

and hence $y' = \frac{1+\pi^2}{2+\pi^2}$ **(2 points)**.

(b) Since $x^{y^2} = y^{x^2}$, we have $e^{y^2 \ln(x)} = e^{x^2 \ln(y)}$, i.e. $y^2 \ln(x) = x^2 \ln(y)$ **(1 point)**. Differentiate the both sides of the equation, we obtain

$$2yy' \ln(x) + y^2/x \ \textbf{(2 point)} = 2x \ln(y) + x^2 y'/y \ \textbf{(2 point)}.$$

After reduction, we conclude that $y' = \frac{2x^2 y \ln(y) - y^3}{2xy^2 \ln(x) - x^3}$ **(1 point)**.

You may also deduce other equivalent answers, e.g. $y' = \frac{2x \ln(y) - y^2/x}{2x \ln(x) - x^2/y}$, or $y' = \frac{y^3(1 - 2\ln(x))}{x^3(1 - 2\ln(y))}$.

5. (12%) Suppose that $f(x)$ is twice differentiable, $\lim_{x\to 1} \frac{(f(x))^3 - 8}{x-1} = 18$, and $\lim_{t\to 0} \frac{f'(1+t) - f'(1-3t)}{t} = 1$.

(a) Find $f(1)$, $f'(1)$ and $f''(1)$.

(b) Suppose that $g(x) = f(e^{2x})$ is a one-to-one function and $h(x) = g^{-1}(x)$, the inverse function of $g(x)$. Find $h(2)$, $h'(2)$ and $h''(2)$.

Solution:

(a) Because $f(x)$ is twice differentiable, $f(x)$ is continuous and $\lim_{x\to 1} f(x) = f(1)$.

Moreover, $\lim_{x\to 1}(f(x))^3 - 8 = \lim_{x\to 1}\left(\frac{(f(x))^3-8}{x-1}\right) \times (x-1) \xlongequal{\text{Limit Law}} \lim_{x\to 1}\left(\frac{(f(x))^3-8}{x-1}\right) \times \lim_{x\to 1}(x-1) = 18 \times 0 = 0$

Hence $\lim_{x\to 1}(f(x))^3 - 8 \xlongequal{\text{Limit Law}} (\lim_{x\to 1} f(x))^3 - 8 = (f(1))^3 - 8 = 0 \Rightarrow f(1) = 2$.(2pts)

$$\begin{aligned}
\lim_{x\to 1}\frac{(f(1))^3-8}{x-1} &= \lim_{x\to 1}\frac{(f(x)-2)[(f(x))^2+2f(x)+4]}{x-1} \\
&= \lim_{x\to 1}[(\frac{f(x)-f(1)}{x-1})\cdot((f(x))^2+2f(x)+4)] \\
&= f'(1)\cdot((f(1))^2+2f(1)+4) \\
&= f'(1)\cdot 12 = 18
\end{aligned}$$

$\Rightarrow f'(1) = \frac{3}{2}$ (2pts)

$$\begin{aligned}
\lim_{t\to 0}\frac{f'(1+t)-f'(1-3t)}{t} &= \lim_{t\to 0}\frac{f'(1+t)-f'(1)+f'(1)-f'(1-3t)}{t} \\
&= \lim_{t\to 0}[\frac{f'(1+t)-f'(1)}{t} + 3\cdot\frac{f'(1)-f'(1-3t)}{3t}] \\
&= f''(1) + 3f''(1) = 4f''(1) = 1
\end{aligned}$$

$\Rightarrow f''(1) = \frac{1}{4}$ (2pts)

(b) $f(1) = 2 \Rightarrow h(2) = g^{-1}(2) = 0$ (2pts)

$g'(x) = f'(e^{2x})\cdot e^{2x}\cdot 2$

$g'(0) = f'(1)\cdot 1\cdot 2 = \frac{3}{2}\cdot 1\cdot 2 = 3$

$g''(x) = f''(e^{2x})\cdot(e^{2x})^2\cdot 2^2 + f'(e^{2x})\cdot e^{2x}\cdot 2^2$

$g''(0) = f''(1)\cdot 1\cdot 4 + f'(1)\cdot 1\cdot 4 = \frac{1}{4}\cdot 1\cdot 4 + \frac{3}{2}\cdot 1\cdot 4 = 7$

We have $g'(h(x))\cdot h'(x) = 1$

then $g''(h(x))\cdot h'(x)\cdot h'(x) + g'(h(x))\cdot h''(x) = 0$

When $x = 2$

$g'(h(2))\cdot h'(2) = 1 \Rightarrow g'(0)\cdot h'(2) = 1$

then $h'(2) = \frac{1}{g'(0)} = \frac{1}{3}$ (2pts)

$g''(h(2))\cdot h'(2)\cdot h'(2) + g'(h(2))\cdot h''(2) = 0$

then $h''(2) = \frac{-g''(0)\cdot h'(2)\cdot h'(2)}{g'(0)} = \frac{-7\cdot\frac{1}{3}\cdot\frac{1}{3}}{3} = -\frac{7}{27}$ (2pts)

6. (10%) Suppose that f is a differentiable function. If $f'(a) > 0$ and $f'(b) < 0$, explain that there exists $c \in (a, b)$ such that $f'(c) = 0$. (Note that f' may not be continuous.)

Solution:

1. (Method 1)
   (1pt) f is differentiable, thus f is continuous.
   (3pt) Since f is continuous on $[a, b]$, by *Extreme Value Theorem* f has some GLOBAL maximum on $[a, b]$. Say one is $f(c)$.
   (2pt) Since $f'(a) > 0$, there exists some $\delta > 0$, $f(a + \delta) > f(a)$. Therefore $f(a)$ is not a local (nor global) maximum. Similarly, $f(b)$ is not, either.
   (4pt) c is not a nor b, then $c \in (a, b)$. By *Fermat's Theorem*, $f'(c) = 0$.

2. (Method 2)
   (1pt) Assume $f(a) = f(b)$, by *Rolle's Theorem* $\exists c \in (a, b)$, $f'(c) = 0$.
   (1pt) Without loss of generality, let $f(a) > f(b)$.
   (2pt) Since $f'(a) > 0$, there exists some $k \in (a, b)$, $f(k) > f(a)$.
   (0pt) Pick some L between $f(k), f(a)$, then L is also between $f(k), f(b)$. By *Intermediate Value Theorem*, there exists some $m \in (a, k)$, $f(m) = L$. Similarly some $n \in (k, b)$, $f(n) = L$.
   (3pt) There exists $m \in (a, k)$, $n \in (k, b)$, $f(m) = f(n)$.
   (3pt) by *Rolle's Theorem* $\exists c \in (m, n) \subset (a, b)$, $f'(c) = 0$.

3. (Method 3)
   (0pt) Since $f'(a) > 0$, there exists some $h_1 > 0$, $f(a+t) > f(a)$ for ALL $0 < t \leq h_1$. Similarly by $f'(b) < 0$, there exists some $h_2 > 0$, $f(b - t) > f(b)$ for all $0 < t \leq h_2$. Take $h = \min\{h_1, h_2, (b - a)/2\}$.
   (2pt) There exists some $h > 0$ satisfies $f(a+h) > f(a)$, $f(b-h) > f(b)$, $a < b-h$. Note that h is a FIXED positive real number here.
   (2pt) Let $F(x) = f(x + h) - f(x)$ on $[a, b - h]$, then $F(x)$ is continuous.
   (3pt) $F(a) > 0$, $F(b - h) < 0$, thus by *Intermediate Value Theorem*, there exists some $d \in (a, b - h)$, $F(d) = 0$.
   (3pt) by *Mean Value Theorem*, there exists some $c \in (d, d+h) \subset (a, b)$, $f'(c) = \dfrac{f(d + h) - f(d)}{(d + h) - d} = \dfrac{F(d)}{h} = 0$.

評分標準:

- There are some proof by contradiction highly related to Method 1 and 2. We omit the proof, and the grading is just the same as Method 1 and 2.
- (NG 1)
  (1pt) f is continuous.
  (2pt) since $f'(a) > 0$, $f'(b) < 0$, there is a local maximum $c \in (a, b)$.
  (4pt) By *Fermat's Theorem*, $f'(c) = 0$.
  Reason: I don't know if you used *Extreme Value Theorem*, *Intermediate Value Theorem* or *It-Looks-Like-That Principle*. Only the first one is correct. This is an example why you are recommended to write down what theorem you used.
- (NG 2)
  (1pt) Assume there does not exist such $c \in (a, b)$, $f'(c) = 0$.
  (0pt) if f is 1-1, then it is strictly increasing (or decreasing) on $[a, b]$. Thus $f'(x) \geq 0$ ($f'(x) \leq 0$) on $[a, b]$, contradicts to $f'(b) < 0$ ($f'(a) > 0$).
  (3pt) if f is not 1-1, then $\exists m, n \in [a, b]$, $m < n$, $f(m) = f(n)$.
  (3pt) Use *Rolle's Theorem* to get a contradiction.
  Reason: "f is 1-1 and continuous $\Longrightarrow$ f is a monotonic function"
  is NON-TRIVIAL. It is more subtle than you think.
- (NG 3)
  (2pt) $f'(a) > 0$ means $\lim\limits_{h\to 0^+} \dfrac{f(a + h) - f(a)}{h} > 0$. Similarly, $f'(b) < 0$ means $\lim\limits_{h\to 0^+} \dfrac{f(b + h) - f(b)}{h} < 0$.
  (2pt) Let $F(x) = f(x + h) - f(x)$, $F(x)$ is continuous.
  (2pt) $F(a) > 0$, $F(b) < 0$, then exists $c \in (a, b)$, $F(c) = 0$.
  (0pt) Then $f'(c) = \lim\limits_{h\to 0^+} F(c)/h = 0$.
  Reason: h is not a fixed number. Then actually we have $F_h(x)$ and c_h depends on h, and there is no reason c_h must converge to a number c.

- If your proof do not match any patterns above, you use the grading below this line. You can get no more than 5pt in total.

- (1pt) f is continuous.
- (4pt) Assume f' is continuous and use *Intermediate Value Theorem*.
- (2pt) Think f' should be continuous and use *Intermediate Value Theorem*, or even not checking continuity first.
- (2pt) Use some other theorem correctly.
- (3pt) Draw a suitable graph of $f(x)$. 1pt for $f'(a) > 0$, 1pt for $f'(b) < 0$, 1pt for horizontal tangent line.
- If even not match any above, you get 1pt for writing something. Only pure empty get 0pt.

Remarks

- The goal of this problem is to prove f' has "intermediate value property", although f' may not be continuous. Note that the *Intermediate Value Theorem* says if f is continuous, then f has this property.
  That f is "smooth" or slope of tangent line of f will change "smoothly" are the consequence of this property, so you cannot use those reason because that is exactly what we are proving.
- Do not use *L'Hôpital's Rule* to prove f' is continuous at x, since you need limit of f' at x exists first, but that is not the case.
- It is good to try proving f' is continuous. But it is not in general...
  See $g(x) = x^2 \sin(\frac{1}{x}), x \neq 0$ and $g(0) = 0$. Then $g(x)$ is differentiable everywhere including 0, while $g'(x)$ is NOT continuous at 0.
- It is good to say if f' is not continuous (but does exist) at x, then f' is oscillating near x. But then you should prove ALL case of oscillating f' have the property, not just draw a single graph for example.
  Actually it is possible to create a everywhere differentiable function f but f' not continuous on all rational number, or even positive measure, which should be not obvious to look on (or draw) the graph.
- We prove "f is 1-1 and continuous on [a, b] $\implies$ f is a monotonic function" here. We need a lemma: if exists $c \in (a, b)$ and $m \in [a, c), n \in (c, b]$, $f(c) \geq f(m)$, $f(c) \geq f(n)$, then f is not 1-1. The proof of the lemma is similar to Method 2 (0pt) part.
  Now we prove the theorem. Without loss of generality let $f(a) < f(b)$, then we need to prove f is strictly increasing. f has global maximum on $[a, b]$, if it is on $c \in (a, b)$, take $m = a, n = b$ and use lemma to get a contradiction. Thus $f(b)$ is the only global maximum, and $f(a)$ is the only global minimum similarly (with a slightly modified lemma). Now if $x_1, x_2 \in (a, b), x_1 < x_2$ but $f(x_1) \geq f(x_2)$, take $c = x_1$, $m = a, n = x_2$ and use lemma to get a contradiction.
  Case (a, b) is similar, for any $x_1, x_2 \in (a, b)$, restrict the domain to $[x_1, x_2]$.

7. (10%) The top of a ladder slides down a vertical wall at a rate of 0.1 m/s. At the moment when the bottom of the ladder is 4 m from the wall, it slides away from the wall at a rate of 0.2 m/s.

   (a) How long is the ladder?

   (b) At that moment, how fast is the angle between the ladder and the ground changing?

Solution:

(a)

1. We are setting that $x(t)$ is the distance between the wall and the buttom of the ladder, and $y(t)$ is the distance between the ground and the top of the ladder. At the moment $t = t_0$, we have $x(t_0) = 4$, $x'(t_0) = 0.2m/s$, and $y'(t_0) = -0.1m/s$.(1%)
   Since $x^2(t) + y^2(t) = L^2$(2%), we have $2x(t)x'(t) + 2y(t)y'(t) = 0$, and it implies $x(t_0)x'(t_0) + y(t_0)y'(t_0) = 0$.(4%)
   Thus, $4 \cdot 0.2 + y(t_0) \cdot (-0.1) = 0 \Rightarrow y(t_0) = 8$.
   Hence the length of the ladder is $L = \sqrt{4^2 + 8^2} = 4\sqrt{5}$(5%)
2. $x = \sqrt{L^2 - y^2}$(1%), so we have $\frac{dx}{dt} = \frac{-y}{\sqrt{L^2 - y^2}}\frac{dy}{dt} = \frac{-\sqrt{L^2 - x^2}}{x}\frac{dy}{dt}$(3%) $\Rightarrow 0.2 = \frac{-\sqrt{L^2 - 16}}{4} \cdot (-0.1)$(4%)
   Thus, $\sqrt{L^2 - 16} = 8 \Rightarrow L = \sqrt{80} = 4\sqrt{5}$.(5%)
3. Assume $x(t) = (4 + 0.2t)$ at $t = 0$. $y = (L^2 - x^2)^{\frac{1}{2}} = (L^2 - (4 + 0.2t)^2)^{\frac{1}{2}}$.(1%)
   $\frac{dy}{dt} = \frac{1}{2}(L^2 - (4 + 0.2t)^2)^{-\frac{1}{2}} \cdot (-2(4 + 0.2t) \cdot 0.2)$(3%)
   Note. $\left.\frac{dy}{dt}\right|_{t=0} = -0.1$
   Thus, at $t = 0$, $-0.1 = \frac{1}{2}(L^2 - 16)^{-\frac{1}{2}} \cdot (-1.6)$(4%) $\Rightarrow \frac{1}{\sqrt{L^2 - 16}} = \frac{1}{8} \Rightarrow L = 4\sqrt{5}$(5%)
4. Solve (b) 4 first. Get $\frac{d\theta}{dt} = \frac{-1}{40}$
   $0.2 = \frac{dL\cos\theta}{dt} = L(-\sin\theta)\frac{d\theta}{dt}$(2%) $= L \cdot \frac{-y}{L} \cdot \frac{-1}{40} \Rightarrow y = 40 \cdot 0.2 = 8$.(4%)
   Thus, $L = \sqrt{4^2 + 8^2} = 4\sqrt{5}$(5%)
5. $y(t) = y_0 - 0.1t$, $x(t) = x_0 + 0.2t = 4 + 0.2t$ as $t \to 0$(1%)
   $\lim_{t\to 0} L^2 = \lim_{t\to 0}(y_0 - 0.1t)^2 + (x_0 + 0.2t)^2$(2%)
   Thus, $\lim_{t\to 0} 0 = \lim_{t\to 0} 2(y_0 - 0.1t)(-0.1) + 2(x_0 + 0.2t)(0.2) = -0.2y_0 + 0.4 \cdot 4 \Rightarrow y_0 = 8$.(4%)
   $L = \sqrt{x_0^2 + y_0^2} = 4\sqrt{5}$(5%)
   Note. Do not assume $y(t) = L - 0.1t$ and $x(t) = 0.2t$.
   $L^2 = (L - 0.1t)^2 + (0.2t)^2 = L^2 - 0.2t + 0.01t^2 + 0.04t^2$ RHS is changed as t is change,but L is constant.$\rightarrow\leftarrow$.
6. Assume $y(t) = L - 0.1t$ $x^2 = L^2 - y^2 = L^2 - L^2 + 0.2Lt - 0.01t^2 = 0.2Lt - 0.01t^2$(1%)
   $x(t) = \sqrt{0.2Lt - 0.01t^2} \Rightarrow x'(t) = \frac{0.1L - 0.01t}{\sqrt{0.2Lt - 0.01t^2}}$(2%)
   When $x(t) = 4$, the behavior of the ladder is same as this assumption.
   $0.2 = x'(t) = \frac{0.1L - 0.01t}{\sqrt{0.2Lt - 0.01t^2}} = \frac{0.1L - 0.01t}{x(t)} = \frac{0.1L - 0.01t}{4} \Rightarrow t = 10L - 80$(4%)
   $16 = x^2 = 0.2Lt - 0.01t^2 = 0.2L(10L - 80) - 0.01(10L - 80)^2 = 2L^2 - 16L - L^2 + 16L - 64 = L^2 - 64 \Rightarrow L = 4\sqrt{5}$(5%)
7. Stand at the top of ladder and see the bottom of ladder.
   During sliding down, the ladder length is same. Thus, We will see the point moving in the circle with radius=L.
   i.e. The velocity v is perpendicular to the radius.(2%)
   θ is the angle between the ladder and the ground.
   $\tan\theta = \frac{0.2}{0.1} = 2 \Rightarrow y = x\tan\theta = 4 \cdot 2 = 8$(4%)
   Thus, $L = 4\sqrt{5}$(5%)

(b) Note. if you write wrong answer in (4) and you use L in (b), you will get at most 4%. Otherwise, if you use cancellation law to delete L, you still can get at most 5%.

1. Since $\cos\theta = \frac{x(t)}{4\sqrt{5}}$(1%), we have $-\sin\theta \cdot \theta'(t) = \frac{x'(t)}{4\sqrt{5}}$.(3%)

   At time $t = t_0$, we know that $\sin\theta = \frac{8}{4\sqrt{5}} = \frac{2}{\sqrt{5}}$ and $x'(t_0) = 0.2$.(4%)

   So $\theta'(t_0) = \frac{x'(t_0)}{4\sqrt{5}} \cdot (-\frac{1}{\sin\theta}) = \frac{0.2}{4\sqrt{5}} \cdot (-\frac{\sqrt{5}}{2}) = \frac{-1}{40}$ (rad/s).(5%)

2. Since $\tan\theta = \frac{y(t)}{x(t)}$(1%), we have $\sec^2\theta \cdot \theta'(t) = \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t)}$.(3%)

   At $t = t_0$, we have $(\sqrt{5}^2) \cdot \theta'(t_0) = \frac{4 \cdot (-0.1) - 8 \cdot (0.2)}{4^2}$(4%) $\Rightarrow \theta'(t_0) = -\frac{1}{8 \cdot 5} = \frac{-1}{40}$(rad/s).(5%)

3. $\theta = \cos^{-1}(\frac{x}{L})$(1%), so $\frac{d\theta}{dt} = \frac{-1}{\sqrt{1 - (\frac{x}{L})^2}} \cdot \frac{\frac{dx}{dt}}{L}$(3%) $= \frac{-1}{\sqrt{1 - \frac{1}{5}}} \cdot \frac{0.2}{4\sqrt{5}} = \frac{-1}{40}$(5%)

4. $-0.1 = \frac{dL\sin\theta}{dt}$(2%) $= L\cos\theta \cdot \frac{d\theta}{dt} = x\frac{d\theta}{dt}$(4%) $= 4\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{-0.1}{4} = \frac{-1}{40}$(5%)

5. In the status of (a) 7.

   $\phi = \frac{\pi}{2} - \theta$(1%). ϕ is the angle between the ladder and the wall

   The velocity v is perpendicular to the radius.

   $v = radius \cdot \frac{d\phi}{dt}$(2%) $= L \cdot \frac{d(\frac{\pi}{2} - \theta)}{dt} = L \cdot (-\frac{d\theta}{dt})$(3%)

   Thus, $0.1\sqrt{5} = \sqrt{0.1^2 + 0.2^2} = v = -4\sqrt{5}\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{-1}{40}$(5%)

8. (12%) A *terminator* (晨昏圈) is a circle that separates the illuminated day side and the dark night side of the Earth. The terminator curve can be characterized by the function $\phi = \tan^{-1}(\cot\phi_0 \cdot \sin\theta)$ on the world map with spherical coordinates $(\theta, \phi) \in [-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where θ is the modified longitude (經度), ϕ is the latitude (緯度), and ϕ_0 is a constant called the declination (赤緯).

Now we focus on the time 4AM, GMT+0, December 2, then $\cot\phi_0 = \sqrt{6}$ and the terminator curve will be

$$\phi = f(\theta) = \tan^{-1}\left(\sqrt{6}\sin\theta\right).$$

The curve is shown in Figure 2. The gray part is at night.

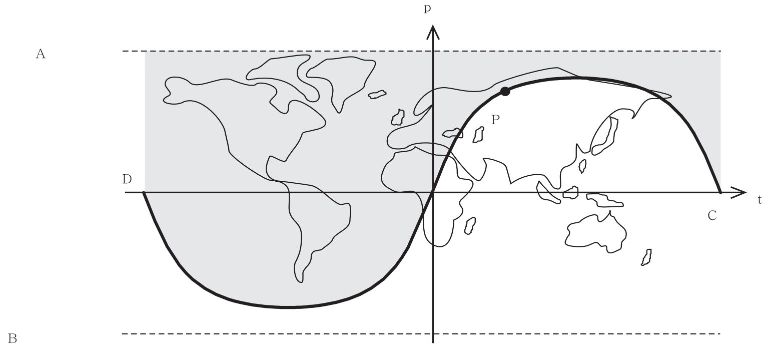

Figure 2: The terminator curve $\phi = f(\theta) = \tan^{-1}\left(\sqrt{6}\sin\theta\right)$ at 4AM, GMT+0, December 2.

(a) City P is located at $P\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$. Find the equation of the tangent line L to the terminator curve at P.

(b) Compute $f''(\theta)$.

(c) City Q is located at $Q\left(\frac{\pi}{3}, \left(\frac{\sqrt{3}}{48} + \frac{1}{3}\right)\pi\right)$. Does City Q lie above, or on, or below the tangent line L?

(d) Is City Q in the daytime or night-time at 4AM, GMT+0, December 2? Explain your answer. (You may use the results of (a),(b), and (c).)

(e) Explain that there must be some place with 9-hours daytime on December 2. (You may observe the regions near the North Pole and South Pole first.)

Solution:

(a) $\frac{d\phi}{d\theta} = \frac{\sqrt{6}\cos\theta}{1 + 6\sin^2\theta}$

$\frac{d\phi}{d\theta}|_{\theta=\frac{\pi}{4}} = \frac{\sqrt{3}}{4}$

The tangent line L is $(\phi - \frac{\pi}{3}) = \frac{\sqrt{3}}{4}(\theta - \frac{\pi}{4})$ (3 pts)

(b) $f''(\theta) = (\frac{\sqrt{6}\cos\theta}{1+6\sin^2\theta})' = \frac{-\sqrt{6}\sin\theta(1+6\sin^2\theta) - (\sqrt{6}\cos\theta)(12\sin\theta\cos\theta)}{(1+6\sin^2\theta)^2}$

$= \frac{-\sqrt{6}\sin\theta - 6\sqrt{6}\sin^3\theta - 12\sqrt{6}\sin\theta\cos^2\theta}{(1+6\sin^2\theta)^2}$ (3 pts)

(c) On L, let $\theta = \frac{\pi}{3} \Rightarrow \phi = \frac{\sqrt{3}}{4}(\frac{\pi}{12}) + \frac{\pi}{3} = \frac{\sqrt{3}\pi}{48} + \frac{\pi}{3}$ Thus Q is on L. (1 pts)

(d) Since $\sin\theta > 0$ on $\theta \in (0, \pi)$ (In particular, $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$) $\Rightarrow f'' < 0$ by (b)
f is concave downward in $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$.(必須說明 f 在這區間凹向下) So point Q is at night time. (2 pts)

(e) Since North Pole has 0-hours daytime on December 2, while South Pole has 24-hours of daytime. And the time of daytime change continuously, there must be a place on the segment joint North Pole and South Pole with 9-hours daytime by Intermediate value Theorem. (3 pts)

9. (14%) Consider the function $f(x) = \dfrac{x^3}{(x+1)^2}$. Answer the following questions by filling each blank below. Show your work (computations and reasoning) in the space following. Put **None** in the blank if the item asked does *not* exist.

(a) The horizontal asymptote of $f(x)$ is: ______________________________.
The vertical asymptote of $f(x)$ is: ______________________________.
The slant asymptote of $f(x)$ is: ______________________________.

(b) $f(x)$ is increasing on the interval(s) ______________________________.
$f(x)$ is decreasing on the interval(s) ______________________________.
Local maximum point(s) of $f(x)$: $(x, y) =$ ______________________________.
Local minimum point(s) of $f(x)$: $(x, y) =$ ______________________________.

(c) $f(x)$ is concave upward on the interval(s) ______________________________.
$f(x)$ is concave downward on the interval(s) ______________________________.
The inflection point(s) $(x, y) =$ ______________________________.

(d) Sketch the graph of $y = f(x)$. Indicate, if any, asymptotes, intervals of increase or decrease, concavity, local extreme values, and points of inflection.

Solution:

(a). $y = \dfrac{x^3}{(x+1)^2} = x - 2 + \dfrac{3x+2}{x^2+2x+1} \to x - 2$ as $x \to \infty$ or $x \to -\infty$, so there is **NO** horizontal asymptote and the slant asymptote $y = x - 2$ exists. For the reason that $\lim_{x \to -1} f(x) = -\infty$, there is a vertical asymptote $x = -1$.

No horizontal asymptote: 1%

Slant asymptote $y = x - 2$: 2%

Vertical asymptote $x = -1$: 1%

(b). $f'(x) = \dfrac{x^3 + 3x^2}{(x+1)^3} = \dfrac{x^2(x+3)}{(x+1)^3}$ (1%)

Increasing interval: $(-\infty, -3] \cup (-1, \infty)$; decreasing interval: $[-3, -1)$. (1%)

Maximum point: $(-3, -27/4)$; minimum point: None. (1%)

(c). $f''(x) = \dfrac{6x}{(x+1)^4}$ (1%)

Concave upward interval: $(0, \infty)$; Concave downward interval: $(-\infty, -1) \cup (-1, 0)$. (1%)

Inflection point: $(0, 0)$.

(d). Asymptotes: 1%

Extreme value and inflection point: 2%

Correct position: 1%