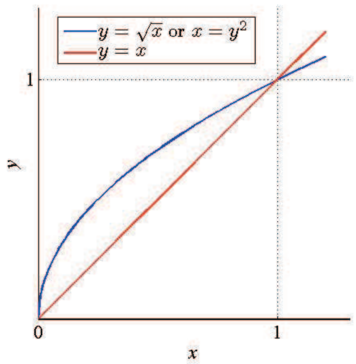
1042微甲06-10班期末考解答和評分標準

1. (10%) Evaluate the iterated integral

$$\int_0^1 \int_x^{\sqrt{x}} \frac{\sin y}{y}\, \mathrm{d}y\, \mathrm{d}x.$$

Solution:



By Fubini's Theorem,

$$\begin{aligned}
&\int_0^1 \int_x^{\sqrt{x}} \frac{\sin y}{y}\, dy dx \\
&= \int_0^1 \int_{y^2}^{y} \frac{\sin y}{y}\, dx dy && (3\%) \\
&= \int_0^1 \frac{\sin y}{y}(y - y^2)\, dy \\
&= \int_0^1 \sin y - y \sin y\, dy && (2\%) \\
&= [-\cos y + y\cos y - \sin y]_0^1 && (3\%) \\
&= [-\cos 1 + \cos 1 - \sin 1] - [-1 + 0 - 0] \\
&= 1 - \sin 1 && (2\%)
\end{aligned}$$

where we use integration by parts for $\int y \sin y\, dy$:

$$\begin{aligned}
u &= y, dv = \sin y\, dy \\
du &= dy, v = -\cos y \\
\int y \sin y\, dy &= -y\cos y + \int \cos y\, dy \\
&= -y \cos y + \sin y + C
\end{aligned}$$

2. (10%) Compute the surface integral

$$\iint_S xz\, \mathrm{d}S,$$

where S is the part of the cone $z = \sqrt{x^2 + y^2}$ inside the circular cylinder $x^2 + y^2 = 2x$.

Solution:

Method 1. The cone can be viewed as a graph of $z = z(x, y) = \sqrt{x^2 + y^2}$. We compute

$$z_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad z_y = \frac{y}{\sqrt{x^2 + z^2}}, \quad \sqrt{1 + z_x^2 + z_y^2} = \sqrt{2}.$$

(以上完成一項得一分 (3%))

So

$$\iint_S xz\,\mathrm{d}S = \iint_D x\sqrt{x^2+y^2}\sqrt{2}\,\mathrm{d}x\,\mathrm{d}z = \sqrt{2}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2\cos\theta}(r\cos\theta\cdot r)r\,\mathrm{d}r\,\mathrm{d}\theta$$

(第一個等式完成得一分；兩個積分範圍上、下限與極坐標面元一項一分 (7%))

$$= \sqrt{2}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\left[\frac{1}{4}r^4\cos\theta\right]\Big|_{r=0}^{r=2\cos\theta}\mathrm{d}\theta = 4\sqrt{2}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\cos^5\theta\,\mathrm{d}\theta$$

(處理到此步驟再得一分 (8%))

$$= 8\sqrt{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}}(1-2\sin^2\theta+\sin^4\theta)\,\mathrm{d}\sin\theta$$

$$= 8\sqrt{2}\left[\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right]\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{64}{15}\sqrt{2}.$$

(處理完成再得兩分 (10%))

Method 2. We parameterize the cone by

$$\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}, \quad 0\le u\le 2\cos v, \quad -\frac{\pi}{2}\le v\le\frac{\pi}{2}.$$

Then we compute

$$\begin{aligned}
\mathbf{r}_u &= \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + 1\,\mathbf{k}\\
\mathbf{r}_v &= -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + 0\,\mathbf{k}\\
\mathbf{r}_u\times\mathbf{r}_v &= -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k}\\
|\mathbf{r}_u\times\mathbf{r}_v| &= \sqrt{(-u\cos v)^2+(-u\sin v)^2+u^2} = \sqrt{2}u.
\end{aligned}$$

(以上完成一項得一分 (4%))

So

$$\iint_S xz\,\mathrm{d}S = \int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\int_{u=0}^{u=2\cos v} u\cos v\cdot u\sqrt{2}u\,\mathrm{d}u\,\mathrm{d}v$$

(兩個積分範圍上、下限與極坐標面元一項一分 (7%))

$$= \sqrt{2}\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\int_{u=0}^{u=2\cos v} u^3\cos v\,\mathrm{d}u\,\mathrm{d}v$$

$$= \sqrt{2}\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\left[\frac{1}{4}u^4\cos\theta\right]\Big|_{u=0}^{u=2\cos v}\mathrm{d}v = 4\sqrt{2}\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\cos^5 v\,\mathrm{d}v$$

(處理到此步驟再得一分 (8%))

$$= 8\sqrt{2}\int_{v=0}^{v=\frac{\pi}{2}}(1-2\sin^2 v+\sin^4 v)\,\mathrm{d}\sin v$$

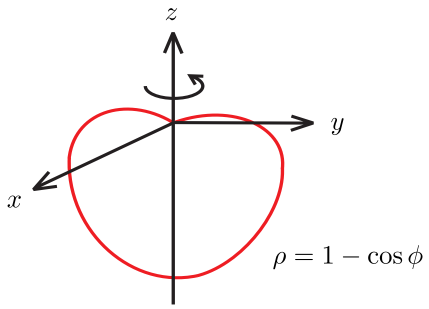
$$= 8\sqrt{2}\left[\sin v-\frac{2}{3}\sin^3 v+\frac{1}{5}\sin^5 v\right]\Big|_{v=0}^{v=\frac{\pi}{2}} = \frac{64}{15}\sqrt{2}.$$

(處理完成再得兩分 (10%))

Remark.

1. 對於 θ 積分範圍從 0 到 π，而且有處理 $\cos^5\theta$ 的積分，最後答案是 0，最多得 7 分。
2. 表面積面元寫成 $\sqrt{5}$，可繼續追究後面的計算，最多給到 5 分。
3. 極坐標若選取 $x = 1 + r\cos\theta, y = r\sin\theta$，將導致無法處理積分，最多給到 5 分。
4. 其他特殊情況，斟酌 0 到 5 分不等。

3. (10%) Find the volume of the cherry, which is enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$.



Solution:

$$\begin{aligned}
\text{volume} &= \iiint_E dV \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi \; d\rho \; d\theta \; d\phi \quad (\mathbf{4pts}) \\
&= \int_0^{\pi}\int_0^{2\pi} \left(\frac{1}{3}\rho^3\Big|_{\rho=0}^{1-\cos\phi} \sin\phi\right) d\theta \; d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \left(\frac{1}{3}(1-\cos\phi)^3 \sin\phi\right) d\theta \; d\phi \quad (\mathbf{2pts}) \\
&= \int_0^{\pi} \left(\frac{1}{3}(1-\cos\phi)^3 \sin\phi\right) d\phi \cdot \int_0^{2\pi} d\theta \\
&= \frac{1}{12}(1-\cos\phi)^4\Big|_{\phi=0}^{\pi} \cdot \theta\Big|_{\theta=0}^{2\pi} \\
&= \frac{8}{3}\pi \quad (\mathbf{4pts})
\end{aligned}$$

4. (13%) Evaluate the double integral $\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dA$, where R is the region bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$ in the first quadrant.

Solution:

第一步：變數變換

[方法1]

- $u = xy, v = \frac{y}{x}$
- $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$
- $1 \le u \le 4, 1 \le v \le 2$

[方法2]

- $u = \sqrt{xy}, v = \sqrt{\frac{y}{x}}$
- $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{2u}{v}$
- $1 \le u \le 2, 1 \le v \le \sqrt{2}$

注：以上三點，每點2分，一共6分；三點任一點出錯則往下不給分。

第二步：寫出積分式

[方法1] $\int\int_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dA = \int_1^4 \int_1^2 \sqrt{v} e^{\sqrt{u}} \cdot \frac{1}{2v} dv du$

[方法2] $\int\int_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dA = \int_1^2 \int_1^{\sqrt{2}} v e^u \cdot |-\frac{2u}{v}| dv du$
注： 第二步2分，若有錯誤往下不給分。

第三步：計算積分
[方法1]

- $\int_1^2 \sqrt{v} \cdot \frac{1}{2v} dv = \sqrt{2} - 1$
- $\int_1^4 e^{\sqrt{u}} du = 2e^2$
- $\int\int_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dA = 2(\sqrt{2}-1)e^2$

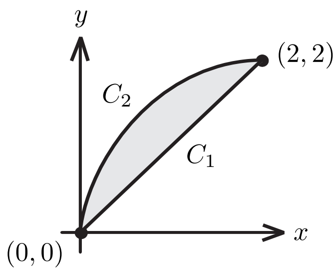
[方法2]

- $\int_1^{\sqrt{2}} v \cdot \frac{2}{v} dv = 2(\sqrt{2}-1)$
- $\int_1^2 u e^{\sqrt{u}} du = e^2$
- $\int\int_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dA = 2(\sqrt{2}-1)e^2$

注：第一點1分，第二點3分，第三點1分，一共五分

5. (14%)

(a) (5%) Evaluate the line integral $I_1 = \int_{C_1} (-\sin x + \mathrm{e}^y)\,\mathrm{d}x + (2x + x\mathrm{e}^y)\,\mathrm{d}y$, where C_1 is the line segment from $(0,0)$ to $(2,2)$.



(b) (4%) Find the area of the region between C_1 and C_2, where C_2 is a curve from $(0,0)$ to $(2,2)$ parameterized by

$$\mathbf{r}(t) = \frac{2}{\pi}(t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}, \quad 0 \le t \le \pi.$$

(c) (5%) Evaluate the line integral $I_2 = \int_{C_2} (-\sin x + \mathrm{e}^y)\,\mathrm{d}x + (2x + x\mathrm{e}^y)\,\mathrm{d}y$.

Solution:

1. Evaluate the line integral $I_1 = \int_{C_1} (-\sin x + e^y)dx + (2x + xe^y)dy$, where C_1 is the line segment from $(0,0)$ to $(2,2)$.
2. Find the area of the region between C_1 and C_2, where C_2 is a curve from $(0,0)$ to $(2,2)$ parameterized by

$$\mathbf{r}(t) = \frac{2}{\pi}(t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \le t \le \pi.$$

3. Evaluate the line integral $I_2 = \int_{C_2} (-\sin x + e^y)dx + (2x + xe^y)dy$.

Solution.

1. We use parametric equation $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, where t from 0 to 2. So $\mathbf{r}'(t) = 1\mathbf{i} + 1\mathbf{j}$, and

$$\begin{aligned} I_1 &= \int_{C_1} (-\sin x + e^y)dx + (2x + xe^y)dy = \int_0^2 (-\sin t + e^t + 2t + te^t)dt \ (3pt) \\ &= \left(\cos t + e^t + t^2 + te^t - e^t\right)\Big|_0^2 = \cos 2 + 2e^2 + 3 \ (2pt) \end{aligned}$$

2. The area is

$$\begin{aligned} \iint_D 1dA &= \frac{2}{\pi}\int_0^\pi (1-\cos t)d(t-\sin t) - \frac{1}{2}\cdot 2\cdot 2 \ (2pt) \\ &= \frac{2}{\pi}\int_0^\pi (1-\cos t)^2 dt - 2 \\ &= \frac{2}{\pi}\int_0^\pi \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right) dt - 2 \\ &= \frac{2}{\pi}\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Big|_0^\pi - 2 = \frac{2}{\pi}\cdot\frac{3}{2}\pi - 2 = 1 \ (2pt) \end{aligned}$$

3. Let $P = -\sin x + e^y$ and $Q = 2x + xe^y$, then $Q_x = 2 + e^y$ and $P_y = e^y$. By Green's Theorem, we have

$$I_1 - I_2 = \int_{C_1 \cup -C_2} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 2dA = 2 \ (4pt)$$

Hence $I_2 = I_1 - 2 = \cos 2 + 1 + 2e^2$ (1pt)

6. (13%)
   (a) (3%) Find the value λ such that the vector field $\mathbf{F} = (x^2 + 4xy^\lambda)\mathbf{i} + (6x^{\lambda-1}y^2 - 2y)\mathbf{j}$ is conservative.
   (b) (5%) For this λ, find a potential function of $\mathbf{F}$.
   (c) (5%) For λ in (a), evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the path described by $\frac{x^2}{9} + (y-1)^2 = 1$ counterclockwise from $(0,0)$ to $(3,1)$.

Solution:

(a) Let $P = x^2 + 4xy^\lambda; Q = 6x^{\lambda-1}y^2 - 2y$. We want to find λ such that F is conservative. We solve the problem:$Q_x = P_y$
$\Rightarrow 6(\lambda-1)x^{\lambda-2}y^2 = 4\lambda xy^{\lambda-1}$
$\Rightarrow \lambda = 3$

(b) $f_x = P; f_y = Q$
$\Rightarrow f = \int x^2 + 4xy^3 dx = \frac{1}{3}x^3 + 2x^2y^3 + g(y)$
$f_y = 6x^2y^2 - 2y = 6x^2y^2 + g'(y), \Rightarrow g(y) = -y^2 + C$
Therefore,$f = \frac{1}{3}x^3 + 2x^2y^3 - y^2 + C$

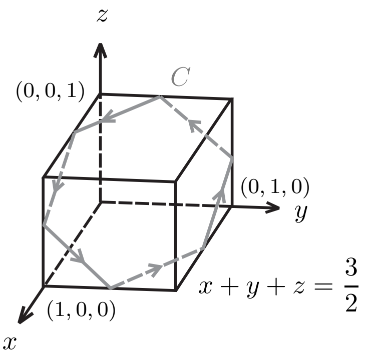
(c) Because F is conservation, $\int_C F\cdot dr$ is independent of path.
Therefore $\int_C F\cdot dr = \int_C \nabla f\cdot dr = f(3,1) - f(0,0) = 26$

7. (15%)
   (a) (3%) Find curl $\mathbf{F}$, where $\mathbf{F}(x,y,z) = (y^2 - z^2)\mathbf{i} + (z^2 - x^2)\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

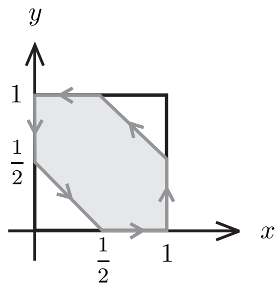
(b) (12%) Compute the line integral

$$\oint_C (y^2 - z^2)\,\mathrm{d}x + (z^2 - x^2)\,\mathrm{d}y + (x^2 - y^2)\,\mathrm{d}z,$$

where C is the hexagon which is the boundary of the intersection of the plane $x + y + z = \frac{3}{2}$ and the unit cube $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$, oriented as pictured.



(a) The curve C in the space.



(b) The curve C projects to xy-plane.

Solution:

(a)

$$curl\vec{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = \langle -2y - 2z, -2x - 2z, -2x - 2y \rangle$$

- 1pt for each component of $curl\vec{F}$.
- If the only definition of $curl\vec{F}$ is right, you can get 1pt.

(b) method 1: Apply Stoke's theorem

$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$$

$$= \oint_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot d\vec{S} \qquad \text{(2pts)}$$

$$= \iint_D -2\langle y + z, x + z, x + y \rangle \cdot \langle 1, 1, 1 \rangle \mathrm{d}x\mathrm{d}y$$

(2pts for D, which is the gray area of image (b).)

(3pts for $d\vec{S}$=$\langle 1, 1, 1 \rangle \mathrm{d}x\mathrm{d}y$))

$$= \iint_D -2 \times 2(x + y + z)\mathrm{d}x\mathrm{d}y = \iint_D -4 \times \frac{3}{2}\mathrm{d}x\mathrm{d}y$$

(If z is replaced by 0, you lose 2pts)

$$= -6 \iint_D \mathrm{d}x\mathrm{d}y$$

$$= -6 \times (1 - \frac{1}{4}) = \frac{-9}{2} \qquad \text{(3pts for the remaining calculations)}$$

method 2: Calculate the line integral directly

$$C_1: r(t) = (\frac{1}{2}+t)\widehat{i} + 0\widehat{j} + (1-t)\widehat{k}, \quad r'(t) = \widehat{i} + 0\widehat{j} - \widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} -(1-t)^2 \mathrm{d}t + 0 + (\frac{1}{2}+t)^2(-\mathrm{d}t) = \int_0^{\frac{1}{2}} (-2t^2 + t - \frac{5}{4})\mathrm{d}t = \frac{-7}{12}$$

$$C_2: r(t) = \widehat{i} + t\widehat{j} + (\frac{1}{2}-t)\widehat{k}, \quad r'(t) = 0\widehat{i} + \widehat{j} - \widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} 0 + ((\frac{1}{2}-t)^2 - 1)\mathrm{d}t + (1-t^2)(-\mathrm{d}t) = \int_0^{\frac{1}{2}} (2t^2 - t - \frac{7}{4})\mathrm{d}t = \frac{-11}{12}$$

$$C_3: r(t) = (1-t)\widehat{i} + (\frac{1}{2}+t)\widehat{j} + 0\widehat{k}, \quad r'(t) = -\widehat{i} + \widehat{j} + 0\widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} (\frac{1}{2}+t)^2(-\mathrm{d}t) - (1-t)^2\mathrm{d}t + 0 = \int_0^{\frac{1}{2}} (-2t^2 + t - \frac{5}{4})\mathrm{d}t = \frac{-7}{12}$$

$$C_4: r(t) = (\frac{1}{2}-t)\widehat{i} + \widehat{j} + t\widehat{k}, \quad r'(t) = -\widehat{i} + 0\widehat{j} + \widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} (1-t^2)(-\mathrm{d}t) + 0 + ((\frac{1}{2}-t)^2 - 1)\mathrm{d}t = \int_0^{\frac{1}{2}} (2t^2 - t - \frac{7}{4})\mathrm{d} = \frac{-11}{12}$$

$$C_5: r(t) = 0\widehat{i} + (1-t)\widehat{j} + (\frac{1}{2}+t)\widehat{k}, \quad r'(t) = 0\widehat{i} - \widehat{j} + \widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} 0 + (\frac{1}{2}+t)^2(-\mathrm{d}t) - (1-t)^2\mathrm{d}t = \int_0^{\frac{1}{2}} (-2t^2 + t - \frac{5}{4})\mathrm{d}t = \frac{-7}{12}$$

$$C_6: r(t) = t\widehat{i} + (\frac{1}{2}-t)\widehat{j} + \widehat{k}, \quad r'(t) = \widehat{i} - \widehat{j} + 0\widehat{k}, \quad 0 \le t \le \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} ((\frac{1}{2}-t)^2 - 1)\mathrm{d}t + (1-t^2)(-\mathrm{d}t) + 0 = \int_0^{\frac{1}{2}} (2t^2 - t - \frac{7}{4})\mathrm{d}t = \frac{-11}{12}$$

So, the line integral is:

$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz = \frac{-7}{12} + \frac{-11}{12} + \frac{-7}{12} + \frac{-11}{12} + \frac{-7}{12} + \frac{-11}{12} = \frac{-9}{2}$$

- 1pt for each parametric equation.
- 1pt for each correct result.

8. (15%)

(a) (3%) Find div $\mathbf{F}$, where $\mathbf{F}(x,y,z) = (y^2 x + \sin z)\mathbf{i} + (x^2 y - \cos x)\mathbf{j} + \left(\frac{1}{3}z^3 + y^2\right)\mathbf{k}$.

(b) (12%) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the top half of the sphere $\begin{cases} x^2+y^2+z^2 = 1 \\ z \ge 0 \end{cases}$ oriented upward.

Solution:

(a)

$divF = y^2 + x^2 + z^2$

(b)

By divergence theorem

$\iint_S F \cdot dS + \iint_{S_2} F \cdot dS = \iiint_{upper\ ball} divF\, dE$ (4pts)

where S_2 is the unit circle in xy-plane with normal vector $(0,0,-1)$

$\iiint_{upper\ ball} divF\, dE = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^2 \rho^2 \sin\phi\, d\phi d\theta d\rho$

$= \frac{1}{5} \cdot 2\pi \cdot -\cos\phi|_{\phi=0}^{\frac{\pi}{2}} = \frac{2\pi}{5}$ (4pts)

$$\iint_{S_2} F \cdot dS = \iint (y^2 x, x^2 y - \cos x, y^2) \cdot (0, 0, -1)\, dA$$
$$= \iint -y^2\, dA = \int_0^{2\pi} \int_0^1 -r^3 \sin^2 \theta\, dr d\theta = -\frac{\pi}{4} \text{ (4pts)}$$
Thus $\iint_S F \cdot dS = \frac{2\pi}{5} - (-\frac{\pi}{4}) = \frac{13}{20}\pi$