1. (8%) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{\sqrt{n}}\right)\ln\left(1+\frac{1}{\sqrt{n}}\right)$ is divergent, conditionally convergent or absolutely convergent.

Solution:

Let $a_n = \sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}})$

Part1:

(1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}}) = 0$ (1pt)

(2) a_n is decreasing (1pt)

Thus the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent by the Alternation Series Test. (2pt)

Part2:

Consider the series $\sum_{n=1}^{\infty}\left|(-1)^n a_n\right| = \sum_{n=1}^{\infty}\sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}})$

We use the Limit Comparison Test with

$$a_n = \sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}}), \quad b_n = \frac{1}{n}$$

and obtain

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\sin(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}}\frac{\ln(1+\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} = 1 \text{ (2pt)}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, the series $\sum_{n=1}^{\infty}\left|(-1)^n a_n\right|$ diverges by the Limit Comparison Test.

Hence the series $\sum_{n=1}^{\infty}(-1)^n\sin(\frac{1}{\sqrt{n}})\ln(1+\frac{1}{\sqrt{n}})$ is conditionally convergent. (2pt)

2. (8%) Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^{4n}}{2n+1}$.

Solution:

Define

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{2n+1}, \quad g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

then $f(0) = 1$ and $f(x) = \frac{1}{x^2} g(x^2)$ for $x \neq 0$.

$g'(x) = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$ for $|x^2| < 1 \Rightarrow |x| < 1$

$g(x) = \int \frac{1}{1-x^2} dx = \frac{1}{2} \int [\frac{1}{1-x} + \frac{1}{1+x}] dx = \frac{1}{2} \ln(\frac{1+x}{1-x}) + C$

By $g(0) = 0$ we know that $C = 0$ such that $g(x) = \frac{1}{2} \ln(\frac{1+x}{1-x})$.

Therefore,

$$f(x) = \begin{cases} \frac{1}{2x^2} \ln(\frac{1+x^2}{1-x^2}) & 0 < |x| < 1 \\ 1 & x = 0 \end{cases}$$

Note that $f(x)$ diverges when $|x| \geq 1$ by the Ratio Test and the Limit Comparison Test with $\sum \frac{1}{n}$ at the end points.

(Another possible answer: since $\int \frac{1}{1-x^2} dx = \tanh^{-1}(x^2) + C$, we also have $f(x) = \frac{1}{x^2} \tanh^{-1}(x^2)$ for $0 < |x| < 1$.)

- Grading policy: 5 points for converting the sum into a function, 3 points for integration.

3. (12%)

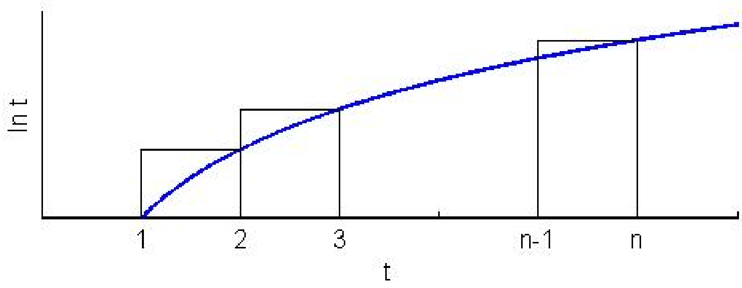
(a) Use a Riemann sum approximation of $\int_1^n \ln t dt$ to show that $\ln(n!) \geq n \ln n - n + 1$.

(b) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$.

Solution:

(5 points for (a), 7 points for (b))

(a) $f(t) = \ln t$ is an increasing function:



From the figure, in $[1, n]$ the upper sum (always taking the value on the right) is larger than the integral. Thus we have $\ln 2 + \ln 3 + \cdots + \ln n \geq \int_1^n \ln t \, dt$ (2 points).

Since $\ln 1 = 0$,

$$\ln(n!) = \ln 1 + \ln 2 + \cdots + \ln n = \ln 2 + \cdots + \ln n \geq \int_1^n \ln t \, dt = t \ln t|_1^n - \int_1^n 1 \, dt = n \ln n - n + 1$$

(3 points)

(b) Define $a_n = \frac{(2n)!}{n^{2n}} x^n$. Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)^{2n+2}}}{\frac{(2n)!}{n^{2n}}} |x| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \left(\frac{n}{n+1}\right)^{2n} |x| = \frac{4}{e^2}|x|$$

in which (by using l'Hospital's Rule)

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} = \exp\left[\lim_{n\to\infty} 2n \ln\left(1 - \frac{1}{n+1}\right)\right] = \exp\left[2 \lim_{n\to\infty} \frac{\frac{\frac{-1}{(n+1)^2}}{1-\frac{1}{n+1}}}{-\frac{1}{n^2}}\right] = e^{-2}.$$

Thus the radius of convergence is $\frac{e^2}{4}$. (4 points)

At $x = \frac{e^2}{4}$, with $\ln(n!)n \geq \ln n - n + 1 \Rightarrow n! \geq \frac{n^n e}{e^n} \Rightarrow (2n)! \geq \frac{(2n)^{2n} e}{e^{2n}}$,

$$a_n = \frac{(2n)!}{n^{2n}} \frac{e^{2n}}{2^{2n}} = (2n)! \frac{e^{2n}}{(2n)^{2n}} \geq \frac{(2n)^{2n} e}{e^{2n}} \frac{e^{2n}}{(2n)^{2n}} = e \neq 0$$

Since $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges by the Test for Divergence.

At $x = -\frac{e^2}{4}$, $\lim_{n\to\infty} a_n$ does not exist (alternating with absolute values larger than e), thus the series also diverges.

In conclusion, the interval of convergence is $(-\frac{e^2}{4}, \frac{e^2}{4})$. (3 points)

4. (8%) Find the Maclaurin series expansion of the function $\ln(1+3x+2x^2)$. Write out the general terms. What is the radius of convergence?

Solution:

Recall that, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $|x| < 1$

$\ln(1+3x+2x^2) = \ln(1+x) + \ln(1+2x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n+1}{n}x^n$

(6 points)

Because the radiu of convergence of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^n}{n}$ is $\frac{1}{2}$, and the radiu of convergence of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x)^n}{n}$ is 1, the radiu of convergence of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n+1}{n}x^n$ is $\frac{1}{2}$ (2 points)

5. (12%)

(a) Find the Maclaurin series for $\sinh^{-1} x$.

(b) Find $\lim_{x\to 0} \frac{\sinh^{-1}(x^2) - x^2}{x^6}$.

Solution:

$(\sinh^{-1}(x))' = (1+x^2)^{\frac{-1}{2}}$

By binomial expansion, $(1+x^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (x^2)^n$

$\binom{\frac{-1}{2}}{n} = \frac{(\frac{-1}{2})(\frac{-3}{2})...(\frac{-1}{2} - n + 1)}{n!} = \frac{(\frac{-1}{2})(\frac{-3}{2})...(-\frac{2n-1}{2})}{n!} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$

We can find the Maclaurin series of

$$\sinh^{-1}(x) = C + \int \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^{2n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$

Because $\sinh^{-1}(0) = 0 \Rightarrow\ C = 0 \quad \Rightarrow$ the Maclaurin series of $\sinh^{-1}(x)$ is $\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$

(8 points)

$\lim_{x\to 0} \frac{\sinh^{-1}(x^2) - x^2}{x^6} = \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{4n+2} - x^2}{x^6}$

$= \lim_{x\to 0} \frac{x^2 + \frac{-1}{6}x^6 + \sum_{n=2}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{4n+2} - x^2}{x^6} = \frac{-1}{6}$ (4 points)

6. (12%) Consider the curve $C: \; x = t^3, \; y = 3t, \; z = t^4$.

(a) Find the curvature of C at the point $(-1, -3, 1)$.

(b) Is there a point on the curve C where the osculating plane is parallel to the plane $x + y + z = 1$?

Solution:

(a) Let $\mathbf{r}(t) = t^3\mathbf{i} + 3t\mathbf{j} + t^4\mathbf{k}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 3\mathbf{j} + 4t^3\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(-1) = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ (1pt)

$\mathbf{r}''(t) = 6t\mathbf{i} + 0\mathbf{j} + 12t^2\mathbf{k} \quad \Rightarrow \quad \mathbf{r}''(-1) = -6\mathbf{i} + 0\mathbf{j} + 12\mathbf{k}$ (1pt)

$\mathbf{r}'(-1) \times \mathbf{r}''(-1) = 36\mathbf{i} - 12\mathbf{j} + 18\mathbf{k}$ (1pt)

$|\mathbf{r}'(-1)| = \sqrt{3^2 + 3^2 + (-4)^2} = \sqrt{34}$ (1pt)

$|\mathbf{r}'(-1) \times \mathbf{r}''(-1)| = \sqrt{36^2 + (-12)^2 + 18^2} = 42$ (1pt)

Hence $\kappa(-1) = \dfrac{|\mathbf{r}'(-1) \times \mathbf{r}''(-1)|}{|\mathbf{r}'(-1)|^3} = \dfrac{21}{17\sqrt{34}} = \dfrac{21\sqrt{34}}{578}$ (1pt)

(b)

$\mathbf{N}(t) = \dfrac{\mathbf{r}''(t)|\mathbf{r}'(t)|^2 - \mathbf{r}'(t)(\mathbf{r}''(t) \cdot \mathbf{r}'(t))}{|\mathbf{r}'(t)|^3}$

$= \mathbf{r}''(t)(\dfrac{1}{|\mathbf{r}'(t)|}) - \mathbf{r}'(t)(\dfrac{\mathbf{r}''(t) \cdot \mathbf{r}'(t)}{|\mathbf{r}'(t)|^3})$ (2pt)

Since $< 1, 1, 1 > \perp \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $< 1, 1, 1 > \perp \mathbf{N}(t)$

$\Rightarrow < 1, 1, 1 > \perp \mathbf{r}'(t)$ and $< 1, 1, 1 > \perp \mathbf{r}''(t)$

$\Rightarrow \begin{cases} < 1,1,1 > \cdot \mathbf{r}'(t) = 0 & \Rightarrow 3t^2 + 3 + 4t^3 = 0 \quad \ldots(1) \\ < 1,1,1 > \cdot \mathbf{r}''(t) = 0 & \Rightarrow 6t + 12t^2 = 0 \quad \ldots(2) \end{cases}$ (1pt)

by (2) we have $t = 0$ or $-\dfrac{1}{2}$ and take it into (1)

$\Rightarrow \begin{cases} 3 \cdot 0 + 3 + 4 \cdot 0 \neq 0 \\ 3 \cdot (-\frac{1}{2})^2 + 3 + 4 \cdot (-\frac{1}{2})^3 \neq 0 \end{cases}$ (1pt)

Hence there is no point on the curve C such that the osculating plane is parallel to the plane $x + y + z = 1$. (2pt)

7. (12%) Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Is f_x continuous at $(0,0)$?

(b) Write down the linear approximation $L(x,y)$ of f at $(0,0)$.

(c) Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}$.

Solution:

(a)

For $(x,y) \neq (0,0)$, $f_x(x,y) = 2x\sin(\dfrac{1}{x^2+y^2}) - \dfrac{2x}{x^2+y^2}\cos(\dfrac{1}{x^2+y^2})$ **(2)**

$$\because \lim_{t\to 0} f_x(t^2,0) = \lim_{t\to 0} 2t^2\sin(\frac{1}{t^4}) - \frac{2t^2}{t^4}\cos(\frac{1}{t^4}) = 0 - \lim_{t\to 0}\frac{2}{t^2}\cos(\frac{1}{t^4})$$
$$= -2\lim_{u\to 0^+}\frac{1}{u}\cos(\frac{1}{u^2}) = -2\lim_{v\to\infty} v\cos(v^2)$$

$\displaystyle\lim_{v\to\infty} v\cos(v^2)$ does not exists.

$\therefore f_x$ is not continuous at $(0,0)$. **(2)**

(b)

$L(x,y) = f(0,0) + f_x(0,0)\Delta x + f_y(0,0)\Delta y$

$$\because f_x(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{t^2\sin(1/t^2)-0}{t} = \lim_{t\to 0} t\sin(\frac{1}{t^2})$$
$= 0$ **(2)**

$$f_y(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{t^2\sin(1/t^2)-0}{t} = \lim_{t\to 0} t\sin(\frac{1}{t^2})$$
$= 0$ **(2)**

$\therefore L(x,y) = 0 + 0\Delta x + 0\Delta y = 0$ **(2)**

(c)

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\sin(\frac{1}{x^2+y^2})$$

Let $x = r\cos\theta$ and $y = r\sin\theta$, then we have:

$$\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\sin(\frac{1}{x^2+y^2}) = \lim_{r\to 0^+}\sqrt{r}\sin(\frac{1}{r})$$

$\because -\sqrt{r} \le \sqrt{r}\sin(\dfrac{1}{r}) \le \sqrt{r}$ and $\displaystyle\lim_{r\to 0+}\sqrt{r} = 0$

$\therefore \displaystyle\lim_{r\to 0^+}\sqrt{r}\sin(\frac{1}{r}) = 0$ by the squeeze theorem

Hence $\displaystyle\lim_{(x,y)\to(0,0)}\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = 0$ **(2)**

8. (12%) Let $f(x,y) = \begin{cases} \dfrac{\sin(x^3) - \sin(y^3)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Calculate $\nabla f(0,0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}} f(0,0)$, where $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$.

(c) Is $f(x,y)$ differentiable at $(0,0)$?

Solution:

(a)

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{\sin(t^3)}{t^3} = 1 \quad \textbf{(3)}$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} -\frac{\sin(t^3)}{t^3} = -1 \quad \textbf{(3)}$$

$$\therefore \nabla f(0,0) = (1,-1)$$

(b)

$$D_u(0,0) = \lim_{t\to 0} \frac{f(0+\frac{1}{\sqrt{2}}t, 0-\frac{1}{\sqrt{2}}t) - f(0,0)}{t} \quad \textbf{(1)}$$

$$= \lim_{t\to 0} \frac{\sin(\frac{t^3}{2\sqrt{2}}) - \sin(-\frac{t^3}{2\sqrt{2}})}{t^3} = \frac{2}{2\sqrt{2}} \lim_{t\to 0} \frac{\sin(\frac{t^3}{2\sqrt{2}})}{\frac{t^3}{2\sqrt{2}}} = \frac{1}{\sqrt{2}} \quad \textbf{(2)}$$

(c)

If $f(x,y)$ is differentiable at $(0,0)$, then $D_u(0,0) = \nabla f(0,0) \cdot u$. **(2)**

However,

$\because D_u(0,0) = \dfrac{1}{\sqrt{2}}$ and $\nabla f(0,0) \cdot u = (1,-1) \cdot (\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}) = \sqrt{2}$

$\therefore D_u(0,0) \neq \nabla f(0,0) \cdot u$, i.e $f(x,y)$ is not differentiable. **(1)**

9. (12%) Suppose that $x,\ y,\ z$ satisfy the relation $x^2+2y^2+3z^2+xy-z=9$. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$.

Solution:

Let $F(x,y,z)=x^2+2y^2+z^2+xy-z-9=0$

Then

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x+y}{6z-1} \text{ (3 points)}$$

And

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{x+4y}{6z-1} \text{ (3 points)}$$

Therefore

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(-\frac{2x+y}{6z-1})=\frac{(-2)(6z-1)+(2x+y)(6z_x)}{(6z-1)^2}=\frac{-2}{6z-1}-6\frac{(2x+y)^2}{(6z-1)^3} \text{ (2 points)}$$

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial}{\partial y}(-\frac{2x+y}{6z-1})=\frac{(-1)(6z-1)+(2x+y)(6z_y)}{(6z-1)^2}=\frac{-1}{6z-1}-6\frac{(2x+y)(x+4y)}{(6z-1)^3} \text{ (2 points)}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(-\frac{x+4y}{6z-1})=\frac{(-4)(6z-1)+(x+4y)(6z_y)}{(6z-1)^2}=\frac{-4}{6z-1}-6\frac{(x+4y)^2}{(6z-1)^3} \text{ (2 points)}$$

10. (12%) Find all critical points of the function $f(x,y) = xye^{-x^2-y^2}$ and classify them.

Solution:

$f_x(x,y) = (1-2x^2)ye^{-x^2-y^2}$ (1 points)

$f_y(x,y) = (1-2y^2)xe^{-x^2-y^2}$ (1 points)

$\Longrightarrow$ critical points are $(0,0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ (2 points)

$f_{xx}(x,y) = (-6xy - 4x^3y)e^{-x^2-y^2}$

$f_{yy}(x,y) = (-6xy - 4xy^3)e^{-x^2-y^2}$

$f_{xy}(x,y) = (1 - 2x^2 - 2y^2 + 4x^2y^2)e^{-x^2-y^2}$ (2 points)

$\Delta(0,0) = -1 < 0$

$\Delta(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{4}{e^2} > 0$ and $f_{xx}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{-2}{e} < 0$

$\Delta(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{4}{e^2} > 0$ and $f_{xx}(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{2}{e} > 0$

$\Delta(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{4}{e^2} > 0$ and $f_{xx}(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{2}{e} > 0$

$\Delta(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{4}{e^2} > 0$ and $f_{xx}(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{-2}{e} < 0$ (4 points)

Hence

$(0,0)$ is a saddle point

$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ are local maximum points

$(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are local minimum points (2 points)

11. (12%) Among all planes that are tangent to the surface $x^2yz = 1$, are there the ones that are nearest or farthest from the origin? Find such tangent planes if they exist.

Solution:

Preliminaries Let $f(x,y,z) = x^2yz$, and let T_r be the tangent plane of a point $r = \langle a,b,c\rangle$ on the surface. The gradient of f on r is

$$\nabla f(r) = \langle 2abc, a^2c, a^2b\rangle = \left\langle \frac{2}{a}, \frac{1}{b}, \frac{1}{c}\right\rangle.$$

Note that here we use the condition $a^2bc = 1$ since r is a point on the surface. Since $\nabla f(r)$ is also the normal vector of T_r, the equation of T_r is

$$\frac{2}{a}x + \frac{1}{b}y + \frac{1}{c}z = 4.$$

The distance between T_r and the origin is

$$d(T_r, 0) = \frac{\left|\frac{2}{a}\cdot 0 + \frac{1}{b}\cdot 0 + \frac{1}{c}\cdot 0 - 4\right|}{\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} = \frac{4}{\sqrt{\frac{4}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}.$$

Now, let

$$g(a,b,c) = \frac{4}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

and find the maxima and minima subject to the constraint $a^2bc = 1$. The maxima of g correspond to the nearest tangent planes, and the minima correspond the farthest. We will use several methods to solve this optimization problem.

Method 1 Applying the AM-GM inequality,

$$g(a,b,c) = \frac{2}{a^2} + \frac{2}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 4\sqrt{\frac{2}{a^2}\cdot\frac{2}{a^2}\cdot\frac{1}{b^2}\cdot\frac{1}{c^2}} = 4\sqrt{\frac{4}{a^4b^2c^2}} = 8.$$

The maxima does not exist since $g \to \infty$ when $a \to 0$. And the minima of g occurs when $2/a^2 = 1/b^2 = 1/c^2$; that is, $a^2 = 2b^2 = 2c^2$. By $a^2bc = 1$, we have

$$a = \pm\sqrt[4]{2} \qquad \text{and} \qquad b = c = \pm\frac{1}{\sqrt[4]{2}}.$$

Method 2 Applying the Lagrange multiplier,

$$\nabla g + \lambda(f - 1) = 0;$$

that is,

$$-\frac{8}{a^3} + \lambda\frac{2}{a} = 0, \qquad -\frac{2}{b^3} + \lambda\frac{1}{b} = 0, \qquad \text{and} \qquad -\frac{2}{c^3} + \lambda\frac{1}{c} = 0.$$

Therefore, the extrema occurs when $\lambda = 4/a^2 = 2/b^2 = 2/c^2$; that is, $a^2 = 2b^2 = 2c^2$. It follows that $g(a,b,c) = 8$. Also, the extrema occurs when the derivative of g does not exist; that is, $a = 0$ or $b = 0$. Since $g \to \infty$ when $a \to 0$ or $b \to 0$, these does are not exist, and we can guarantee that those extrema with $g = 8$ are global minima.

Method 3 Replacing c by $1/a^2b$,

$$g(a,b) = \frac{4}{a^2} + \frac{1}{b^2} + a^4b^2.$$

The first order partial derivatives are

$$g_a = -\frac{8}{a^3} + 4a^3b^2 \qquad \text{and} \qquad g_b = -\frac{2}{b^3} + 2a^4b,$$

and the second order partial derivatives are

$$g_{aa} = \frac{24}{a^4} + 12a^2b^2, \qquad g_{ab} = g_{ba} = 8a^3b, \qquad \text{and} \qquad g_{bb} = \frac{6}{b^4} + 2a^4.$$

Therefore, the extrema occurs when $g_a = 0$ and $g_b = 0$; that is,

$$a = \pm\sqrt[4]{2} \qquad \text{and} \qquad b = c = \pm\frac{1}{\sqrt[4]{2}}.$$

It follows that

$$D = g_{aa}g_{bb} - g_{ab}^2 = 24 \cdot 16 - \left(\pm 8\sqrt{2}\right) = 384 - 128 = 256 > 0;$$

that is, these extrema are local minima. Also, the extrema occurs when derivative of g does not exist; that is, $a = 0$ or $b = 0$. Since $g \to \infty$ when $a \to 0$ or $b \to 0$, these does are not exist, and we can guarantee that those local minima are global minima.

Results After solving the optimization problem, we find the farthest tangent planes

$$\begin{aligned}
2^{3/4}x + 2^{1/4}y + 2^{1/4}z &= 1,\\
2^{3/4}x - 2^{1/4}y - 2^{1/4}z &= 1,\\
-2^{3/4}x + 2^{1/4}y + 2^{1/4}z &= 1,\\
-2^{3/4}x - 2^{1/4}y - 2^{1/4}z &= 1.
\end{aligned}$$

The nearest tangent plane does not exist since g has no maxima.

Points

- (2%) Find $\nabla f(r)$.
- (2%) Find equation of T_r.
- (2%) Find the distance between T_r and the origin.
- (2%) Find the extrema of g.
- (2%) Find farthest tangent planes.
- (2%) Show that the nearest tangent plane does not exist.