1041微乙01-05班期末考解答和評分標準

1. (15%) 令 $f(x) = \int_x^{x^2} \frac{1}{1+t^4}\, dt$ 且 $F(x) = xe^{f(x)}$。
   (a) (10%) 求 $f'(x)$。
   (b) (5%) 求 $F'(1)$。

Solution:
(By Fundamental Theorem of Calculus)

$$f'(x) = \frac{2x}{1+x^8} - \frac{1}{1+x^4}$$

Problem 1(b): $F(x) = xe^{f(x)}$. Find $F'(1)$.

Solution :

$$\begin{aligned} F'(x) &= e^{f(x)} + xe^{f(x)} f'(x) \\ F'(1) &= e^{f(1)} + e^{f(1)} f'(1) \\ &= \frac{3}{2} \end{aligned}$$

where $f(1) = 0$ and $f'(1) = \frac{1}{2}$

2. (15%) 令 $F(x) = \int_0^x e^{-t^2}\,dt$。求 $F(x)$ 在 $x = 0$ 的泰勒展開式並寫出一般項。

Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots \quad (4pts) \\ &= \sum_{k=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

$$\begin{aligned} F'(x) &= e^{-x^2} \\ &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \ldots + \frac{(-x^2)^n}{n!} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \quad (6pts) \end{aligned}$$

$$\begin{aligned} F(x) &= \int_0^x e^{-x^2} dx \\ &= \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(n!)} + C \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(n!)} \qquad (F(0) = 0 \Rightarrow C = 0) \quad (5pts) \end{aligned}$$

若只求出泰勒展開式的正確前4項以上，而沒求出一般項者，最多可拿7分。

3. (10%) 計算積分值: $\int_{-1}^{0} \frac{x}{\sqrt{3-2x-x^2}}\ dx.$

Solution:

$\int_{-1}^{0} \frac{x}{\sqrt{4-(x+1)^2}}dx = \frac{1}{2}\int_{-1}^{0} \frac{x}{\sqrt{1-(\frac{(x+1)^2}{2})}}$

let $\frac{x+1}{2} = \sin\theta$

$\frac{dx}{2} = \cos\theta d\theta$

when $x = -1 \quad \theta = 0 \quad x = 1 \quad \theta = \frac{\pi}{6}$(5pts)

$\int_{0}^{\frac{\pi}{6}} \frac{2\sin\theta - 1}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$

$\int_{0}^{\frac{\pi}{6}} (2\sin\theta - 1)d\theta$

$= 2 * (-\frac{\sqrt{3}}{2} + 1) - \frac{\pi}{6}$

$= 2 - \sqrt{3} - \frac{\pi}{6}$ (10pts)

4. (15%) 求積分值: $\int_1^2 \frac{x^4+x^2-1}{x^3+x}\,dx.$

Solution:

$\int_1^2 \frac{x^4+x^2-1}{x^3+x}dx = \int_1^2 (x - \frac{1}{x^3+x})dx \quad -(*)$

$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$ (5pts)

$(*) = \int_1^2 (x - \frac{1}{x} + \frac{x}{x^2+1})dx$ (10pts)

$= \int_1^2 (x - \frac{1}{x})dx + \int_1^2 \frac{d(x^2+1)}{2(x^2+1)}$

$\frac{x^2}{2} - ln(x) + \frac{ln(x^2+1)}{2} \,|_1^2$

$= \frac{3}{2} - \frac{3ln2}{2} + \frac{ln5}{2}$ (15pts)

5. (10%) 求極限值: $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x$.

Solution:

$\lim_{x\to\infty}(1+\sin\frac{3}{x})^x = \lim_{x\to\infty} e^{x\ln(1+\sin\frac{3}{x})}$ (3 pts)

$$= e^{\lim_{x\to\infty}\frac{\ln(1+\sin\frac{3}{x})}{\frac{1}{x}}(\frac{0}{0})} \ (1pt)$$

(L'Hospital Rule)

$$= e^{\lim_{x\to\infty}\frac{\frac{1}{1+\sin\frac{3}{x}}(\cos\frac{3}{x})(\frac{-3}{x^2})}{\frac{-1}{x^2}}} \ (4pts)$$

$$= e^{\lim_{x\to\infty}\frac{1}{1+\sin\frac{3}{x}}(\cos\frac{3}{x})(3)} = e^3 \ (2pts)$$

6. (10%) 求由曲線 $x^4 - x^2 - y^2 = 0$ 和 $x = 2$ 所圍成區域的面積。

Solution:

$y^2 = x^2(x^2 - 1)(|x| \geq 1)$ (2 pts)

So, the area is $2\int_1^2 \sqrt{x^4 - x^2}dx$ (4 pts)

$2\int_1^2 \sqrt{x^4 - x^2}dx = 2\int_1^2 x\sqrt{x^2 - 1}dx = \int_1^2 \sqrt{x^2 - 1}d(x^2 - 1)$ (2 pts)

$= \frac{2}{3}(x^2 - 1)^{\frac{3}{2}} |_1^2 = 2\sqrt{3}$ (2 pts)

Note that you can get 2 points if you only sketched the graph or find the intersection.

7. (15%) 設 R 為 $y=\cos\left(\frac{\pi}{2}x\right)$, $y=0$ 及 $0\le x\le 1$ 所圍成的區域。

(a) (8%) 求 R 繞 x 軸旋轉所產生之體積。

(b) (7%) 求 R 繞 y 軸旋轉所產生之體積。

Solution:

(a) 有寫 $\int \pi f^2(x)dx$ (2分)

方法一

$$\begin{aligned}
&\int_0^1 \pi\cos^2(\frac{\pi}{2}x)dx \quad (3分)\\
&=\pi\int_0^1 \frac{\cos\pi x+1}{2}dx \quad (2分)\\
&=\frac{\pi}{2}+\frac{\pi}{2}\int_0^1 \cos\pi x dx\\
&=\frac{\pi}{2}+\frac{1}{2}\sin\pi x|_0^1 \quad (2分)\\
&=\frac{\pi}{2} \quad (1分)
\end{aligned}$$

方法二

$$\begin{aligned}
&\int_0^1 \pi\cos^2(\frac{\pi}{2}x)dx \quad (3分)\\
&\text{Let } t=\frac{\pi}{2}x,\ dt=\frac{\pi}{2}dx \quad (2分)\\
&=\frac{2}{\pi}\cdot\pi\int_0^{\frac{\pi}{2}}\cos^2 tdt\\
&=2\int_0^{\frac{\pi}{2}}\frac{\cos 2t+1}{2}dt \quad (2\ 分)\\
&=\int_0^{\frac{\pi}{2}}\cos 2tdt+\frac{\pi}{2}\\
&=\frac{1}{2}\sin 2t|_0^{\frac{\pi}{2}}+\frac{\pi}{2}\\
&=\frac{\pi}{2} \quad (1分)
\end{aligned}$$

(b) 有寫 $\int 2\pi x f(x)dx$ (2分)

方法一

$$\begin{aligned}
&\int_0^1 2\pi x\cos(\frac{\pi}{2}x)dx \quad (3分)\\
&=2\pi\int_0^1 x\cdot\frac{2}{\pi}d(\sin\frac{\pi}{2}x)\\
&=4x\sin(\frac{\pi}{2}x)|_0^1-4\int_0^1\sin\frac{\pi}{2}xdx \quad (2分)\\
&=4+4\cdot\frac{2}{\pi}\cos\frac{\pi}{2}x|_0^1 \quad (1\ 分)\\
&=4-\frac{8}{\pi} \quad (1分)
\end{aligned}$$

方法二

$$\begin{aligned}
&\int_0^1 2\pi x\cos(\frac{\pi}{2}x)dx \quad (3分)\\
&\text{Let } t=\frac{\pi}{2}x,\ dt=\frac{\pi}{2}dx \quad (2分)\\
&=2\pi\int_0^{\frac{\pi}{2}}(\frac{2}{\pi}t)\cos t\cdot\frac{2}{\pi}dt\\
&=\frac{8}{\pi}\int_0^{\frac{\pi}{2}}t\cos tdt\\
&=\frac{8}{\pi}(t\sin t|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin tdt)=\frac{8}{\pi}(\frac{\pi}{2}+\cos t|_0^{\frac{\pi}{2}})=4-\frac{8}{\pi} \quad (2分)
\end{aligned}$$

8. (10%) 求曲線 $y = \frac{e^x + e^{-x}}{2}$, 由 $x = -1$ 到 $x = 1$ 之長度。

Solution:

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

$$L = \int_{-1}^{1} \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_{-1}^{1} \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} dx$$

$$= \int_{-1}^{1} \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} dx = \int_{-1}^{1} \frac{e^x + e^{-x}}{2} dx$$

$$= \frac{1}{2}(e^x|_{-1}^{1} - e^{-x}|_{-1}^{1}) = e - \frac{1}{e}$$

評分標準:
寫出長度積分公式 (3%) 積分上下限正確(1%)
y 的微分正確 (2%)
積分長度公式 (4%)