## 1031微甲07-11班期末考解答和評分標準

1. (10%) Solve the initial value problem: $\dfrac{dy}{dx} = \dfrac{e^y}{1+x^2}$, $y(0) = -1$.

**Solution:**

$$\frac{dy}{dx} = \frac{e^y}{1+x^2} \text{ (separable)}$$
$$\Rightarrow e^{-y}dy = \frac{1}{1+x^2}dx \text{ (2\%)}$$
$$\Rightarrow -e^{-y} = \tan^{-1}x + C \text{ (3\% for each side)}$$

With $y(0) = -1$, we have $-e = \tan^{-1}0 + C = C$.
$\Rightarrow C = -e$ (2%)
$\therefore$ The solution is $e^{-y} = e - \tan^{-1}x$

2. (a) (10%) Solve the initial value problem:

$$\begin{cases} 2x(x+3)y' + (4x+3)\,y = 2x^{\frac{1}{2}}(x+3)^{\frac{1}{2}}, \\ y(1) = \dfrac{1}{2}, \quad x > 0. \end{cases}$$

(b) (3%) Find $\lim\limits_{x\to\infty} y(x)$ and $\lim\limits_{x\to 0^+} y(x)$.

**Solution:**

(a)

Divide the equation by $2x\,(x+3)$

We get $y' + \dfrac{4x+3}{2x(x+3)}y = \dfrac{1}{\sqrt{x(x+3)}}$

Multiply integration factor $I(x)$ on both side, then

$Iy' + I\dfrac{4x+3}{2x(x+3)}y = I\dfrac{1}{\sqrt{x(x+3)}}$

Solve that $I' = I\dfrac{4x+3}{2x(x+3)}$, we let $I = e^{\int \frac{4x+3}{2x(x+3)}dx}$

Where $\displaystyle\int \frac{4x+3}{2x(x+3)}dx = \frac{1}{2}\int(\frac{1}{x} + \frac{3}{x+3})dx = \frac{1}{2}\ln x + \frac{3}{2}\ln(x+3) + C$, when $x > 0$

Hence $I(x) = e^C\sqrt{x(x+3)^3}$, Let $C = 0 \Rightarrow I(x) = \sqrt{x(x+3)^3}$

$Iy' + I\dfrac{4x+3}{2x(x+3)}y = Iy' + I'y = (Iy)' = I\dfrac{1}{\sqrt{x(x+3)}} = x+3$

$Iy = \displaystyle\int(x+3)dx = \frac{1}{2}x^3 + 3x + D \Rightarrow \sqrt{x(x+3)^3}y = \frac{1}{2}x^3 + 3x + D$

Bring $y(1) = \dfrac{1}{2}$ into the equation, we find $D = \dfrac{1}{2}$

Hence $y = \dfrac{x^2+6x+1}{2\sqrt{x(x+3)^3}}$

(b)

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty}\sqrt{\dfrac{(1+\frac{6}{x}+\frac{1}{x^2})^2}{4(1+\frac{3}{x})^3}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

Because $\lim\limits_{x\to 0^+} x^2+6x+1 = 1$, and $\lim\limits_{x\to 0^+} 2\sqrt{x(x+3)^3} = 0$

So $\lim\limits_{x\to 0^+} y \to +\infty$

評分標準如下，

(a)

解出 $I(x)$ +5分

解出$y$ 的不定積分 +3分

代入初始條件求出完整解答 +2分

計算錯誤該部分加分折半

(b)
(a)解出的 $y$ 錯誤不管答案一律0分
(有些同學計算錯誤會影響答案，有些不會，覺得不應該同樣因為計算錯誤而有差別)
此外對一題2分，兩題3分

3. (a) (7%) Find the length of the curve in polar coordinates: $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le 2\pi$.
   (b) (7%) Find the area enclosed by the curve given in (a).

**Solution:**

(a)

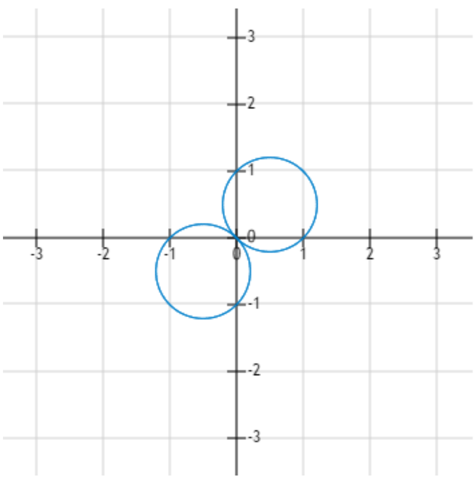
$$\frac{dr}{d\theta} = \frac{d}{d\theta}\sqrt{1+\sin 2\theta} = \frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}} \qquad \text{(2 points)}$$

The length of the curve $L$ is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta && \text{(2 points)} \\ &= \int_0^{2\pi} \sqrt{1 + \sin 2\theta + \frac{\cos^2 2\theta}{1+\sin 2\theta}}\, d\theta \\ &= \int_0^{2\pi} \sqrt{\frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1+\sin 2\theta}}\, d\theta \\ &= \int_0^{2\pi} \sqrt{\frac{2 + 2\sin 2\theta}{1+\sin 2\theta}}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2}\, d\theta \\ &= 2\sqrt{2}\pi && \text{(3 points)} \end{aligned}$$

(b) Because $r \ge 0$ for $0 \le \theta \le 2\pi$ (or due to observation from the figure below), the area $A$ of the enclosed region is

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2\, d\theta && \text{(4 points)} \\ &= \frac{1}{2}\int_0^{2\pi} (1 + \sin 2\theta)\, d\theta \\ &= \frac{1}{2}\left[\theta - \frac{1}{2}\cos 2\theta\right]_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi - \frac{1}{2} - 0 + \frac{1}{2}\right) \\ &= \pi && \text{(3 points)} \end{aligned}$$



4. (a) (7%) Find the length of the parametric curve $C$: $x = \ln(\sec t + \tan t) - \sin t,\ y = \cos t,\ \ 0 \le t \le \frac{\pi}{3}$.

(b) (7%) Rotate the curve $C$ about $x$-axis. Find the surface area.

(c) (7%) Find the volume of the solid bounded by the surface given in (b) and the planes $x = 0$, $x = \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$.

**Solution:**

(a).

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)) = sec(t) - cos(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}(y(t)) = -sin(t). \qquad \ldots(1 \quad points)$$

$$\begin{aligned}\int_0^{\frac{\pi}{3}} ds &= \int_0^{\frac{\pi}{3}} tan(t)dt \qquad \ldots(2 \quad points)\\ &= -\ln|cos(t)|_0^{\pi/3} \qquad \ldots(2 \quad points)\\ &= \ln(2) \quad \sharp \qquad \ldots(2 \quad points)\end{aligned}$$

, where $ds = \sqrt{\left[\frac{\mathrm{d}}{\mathrm{d}t}(x(t))\right]^2 + \left[\frac{\mathrm{d}}{\mathrm{d}t}(y(t))\right]^2}\,dt = tan(t)dt.$

(b).

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\pi y(t)ds &= \int_0^{\frac{\pi}{3}} 2\pi con(t)tan(t)dt \qquad \ldots(2 \quad points)\\ &= -2\pi cos(t)|_0^{\pi/3} \qquad \ldots(3 \quad points)\\ &= \pi \quad \sharp \qquad \ldots(2 \quad points)\end{aligned}$$

(c).

$$\begin{aligned}\int \pi y^2(t)d(x(t)) &= \int_0^{\frac{\pi}{3}} \pi con^2(t)[sec(t) - cos(t)]dt \qquad \ldots(2 \quad points)\\ &= \int_0^{\frac{\pi}{3}} \pi sin^2(t)con(t)dt\\ &= \frac{\pi}{3} sin^3(t)|_0^{\pi/3} \qquad \ldots(3 \quad points)\\ &= \frac{\sqrt{3}}{8}\pi \quad \sharp \qquad \ldots(2 \quad points)\end{aligned}$$

, where $d(x(t)) = [sec(t) - cos(t)]dt$.

5. (10%) Find $\int \frac{1}{x^2(x^2+x+1)}dx$.

**Solution:**

Let $\frac{1}{x^2(x^2+x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx+d}{x^2+x+1}$, where $a, b, c, d$ are determined. (1 point) Then solve $a = -1$,$b = 1$,$c = 1$,$d = 0$. (2 points, if solve partially 1 point)

so we have

$$\int \frac{1}{x^2(x^2+x+1)}dx = \int \frac{-1}{x}dx + \int \frac{1}{x^2}dx + \int \frac{x}{x^2+x+1}dx$$

$= -\ln|x| - \frac{1}{x} + \int \frac{x}{x^2+x+1}dx$ (2 points)

$\int \frac{x}{x^2+x+1}dx = \int \frac{x+\frac{1}{2}}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx = \frac{1}{2}\ln(x^2+x+1)$ (2 points)

$\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} = \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$(3 points)

so Answer is $-\ln|x| - \frac{1}{x} + \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$

If you don't wirte $+C$, you will lose 1 point.

6. (10%) Find $\int \frac{1}{\sec x - 1}dx$.

**Solution:**

**Solution 1.**

$$\begin{aligned}\int \frac{1}{\sec x-1}\,\mathrm{d}x &= \int \frac{1}{\sec x-1}\cdot\frac{\sec x+1}{\sec x+1}\,\mathrm{d}x = \int \frac{\sec x+1}{\sec^2 x-1}\,\mathrm{d}x \ \textit{(2分)}\\ &= \int \frac{\sec x+1}{\tan^2 x}\,\mathrm{d}x \ \textit{(3分)} = \int\left(\frac{\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right)\mathrm{d}x\\ &= \int \frac{1}{\sin^2 x}\,\mathrm{d}\sin x + \int \frac{1-\sin^2 x}{\sin^2 x}\,\mathrm{d}x\\ &= \int \frac{1}{\sin^2 x}\,\mathrm{d}\sin x + \int \csc^2 x\,\mathrm{d}x - \int 1\,\mathrm{d}x\\ &= -\frac{1}{\sin x} - \cot x - x + C.\end{aligned}$$

後面的三項積分個別計分，第一項積出 $-\frac{1}{\sin x}$ 再得 *4* 分，其它兩項及常數 $C$ 一共佔 *3* 分。

**Solution 2.** *Let* $t = \tan\frac{x}{2}$ *(佔 1 分), then we have*

$$\cos x = \frac{1-t^2}{1+t^2} \ \textit{(佔 1 分)},\ \sin x = \frac{2t}{1+t^2} \ \textit{(沒有用到)},\ \textit{and}\ \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.\ \textit{(佔 1 分)}$$

*So*

$$\begin{aligned}\int \frac{1}{\sec x-1}\,\mathrm{d}x &= \int \frac{\cos x}{1-\cos x}\,\mathrm{d}x \ \textit{(2分)} = \int \frac{\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}\,\mathrm{d}t = \int \frac{1-t^2}{t^2(1+t^2)}\,\mathrm{d}t \ \textit{(5分)}\\ &= \int\left(\frac{1}{t^2}-\frac{2}{t^2+1}\right)\mathrm{d}t \ \textit{(7分)} = -\frac{1}{t} - 2\tan^{-1}t + C \ \textit{(9分)}\\ &= -\frac{1}{\tan\frac{x}{2}} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C = -\cot\frac{x}{2} - x + C.\ \textit{(10分)}\end{aligned}$$

前面兩種解法是大部分人採用的積分策略，所以在此列出詳細配分標準，以下還有數種解法，處理方式都很漂亮，在此也完整呈現供大家參考及學習，配分以個案處理， 但原則上和前兩種配分方式雷同。

**Solution 3.**

$$\begin{aligned}\int \frac{1}{\sec x-1}\,\mathrm{d}x &= \int \frac{\cos x}{1-\cos x}\,\mathrm{d}x = \int \frac{1-2\sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}}\,\mathrm{d}x\\ &= \int \csc^2\frac{x}{2}\,\mathrm{d}\left(\frac{x}{2}\right) - \int 1\,\mathrm{d}x = -\cot\frac{x}{2} - x + C.\end{aligned}$$

**Solution 4.**

$$\int \frac{1}{\sec x-1}\,\mathrm{d}x = \int \frac{1}{\sec x-1}\cdot\frac{\sec x+1}{\sec x+1}\,\mathrm{d}x = \int \frac{\sec x+1}{\sec^2 x-1}\,\mathrm{d}x = \int \frac{\sec x+1}{\tan^2 x}\,\mathrm{d}x.$$

*Since*

$$\int \frac{\sec x}{\tan^2 x}\,\mathrm{d}x = \int \frac{\sec x\tan x}{\tan^3 x}\,\mathrm{d}x = \int \frac{1}{\tan^3 x}\,\mathrm{d}\sec x = \frac{\sec x}{\tan^3 x} - \int \sec x\,\mathrm{d}\cot^3 x$$
$$= \frac{\sec x}{\tan^3 x} + \int 3\sec x\cot^2 x\csc^2 x\,\mathrm{d}x = \frac{\sec x}{\tan^3 x} + 3\int \frac{\cos x}{\sin^4 x}\,\mathrm{d}x$$
$$= \frac{\sec x}{\tan^3 x} + 3\int \frac{1}{\sin^4 x}\,\mathrm{d}\sin x = \frac{\sec x}{\tan^3 x} - \frac{1}{\sin^3 x} + C,$$

*and*

$$\int \frac{1}{\tan^2 x}\,\mathrm{d}x = \int \cot^2 x\,\mathrm{d}x = \int (\csc^2 x - 1)\,\mathrm{d}x = -\cot x - x + C,$$

*we get*

$$\int \frac{1}{\sec x - 1}\,\mathrm{d}x = \frac{\sec x}{\tan^3 x} - \frac{1}{\sin^3 x} - \cot x - x + C.$$

**Solution 5.**

$$\int \frac{1}{\sec x - 1}\,\mathrm{d}x = \int \frac{1}{\sec x - 1}\cdot\frac{\sec x + 1}{\sec x + 1}\,\mathrm{d}x = \int \frac{\sec x + 1}{\sec^2 x - 1}\,\mathrm{d}x = \int \frac{\sec x + 1}{\tan^2 x}\,\mathrm{d}x.$$

*Since*

$$\int \frac{\sec x}{\tan^2 x}\,\mathrm{d}x = \int \cot x\csc x\,\mathrm{d}x = -\csc x + C,$$

*and*

$$\int \frac{1}{\tan^2 x}\,\mathrm{d}x = \int \frac{\cos^2 x}{\sin^2 x}\,\mathrm{d}x = \int \cos^2 x\csc^2 x\,\mathrm{d}x = -\int \cos^2 x\,\mathrm{d}\cot x$$
$$= -\cos^2 x\cot x + \int \cot x\,\mathrm{d}\cos^2 x$$
$$= -\cos^2 x\cot x - 2\int \cot x\cos x\sin x\,\mathrm{d}x$$
$$= -\cos^2 x\cot x - 2\int \cos^2 x\,\mathrm{d}x = -\cos^2 x\cot x - \int (1+\cos 2x)\,\mathrm{d}x$$
$$= -\cos^2 x\cot x - x - \frac{\sin 2x}{2} + C$$
$$-\cos^2 x\cot x - x - \sin x\cos x + C$$

*we get*

$$\int \frac{1}{\sec x - 1}\,\mathrm{d}x = -\csc x - \cos^2 x\cot x - x - \sin x\cos x + C.$$

**Solution 6.** *Let* $t = \cot\frac{x}{2}$*, then we have*

$$\cos x = \frac{t^2-1}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \textit{and} \quad \mathrm{d}x = -\frac{2}{1+t^2}\,\mathrm{d}t.$$

*So*

$$\int \frac{1}{\sec x - 1}\,\mathrm{d}x = \int \frac{\cos x}{1-\cos x}\,\mathrm{d}x = \int \frac{\frac{t^2-1}{1+t^2}\cdot\frac{-2}{1+t^2}}{1-\frac{t^2-1}{1+t^2}}\,\mathrm{d}t = \int \frac{1-t^2}{1+t^2}\,\mathrm{d}t$$
$$= \int \left(-1 + \frac{2}{1+t^2}\right)\mathrm{d}t = -t + 2\tan^{-1} t + C$$
$$= -\cot\frac{x}{2} + 2\tan^{-1}\left(\cot\frac{x}{2}\right) + C.$$

**Solution 7.** *Let* $t = \sec x + \tan x$*. Since* $\sec^2 x - \tan^2 x = (\sec x + \tan x)(\sec x - \tan x) = 1$*, we have* $\frac{1}{t} = \sec x - \tan x$ *and*

$$\sec x = \frac{t^2+1}{2t}, \quad \tan x = \frac{t^2-1}{2t}, \quad \mathrm{d}t = \frac{t^2+1}{2}\,\mathrm{d}x.$$

*So*

$$\int \frac{1}{\sec x - 1}\,\mathrm{d}x = \int \frac{1}{\frac{t^2+1}{2t} - 1}\cdot\frac{2}{t^2+1}\,\mathrm{d}t = \int \frac{4t}{(t-1)^2(t^2+1)}\,\mathrm{d}t$$
$$= \int \left(\frac{2}{(t-1)^2} - \frac{2}{t^2+1}\right)\mathrm{d}t = -\frac{2}{t-1} - 2\tan^{-1} t + C$$
$$= -\frac{2}{\sec x + \tan x - 1} - 2\tan^{-1}(\sec x + \tan x) + C.$$

7. Consider the intersection of three circular cylinders with radius $R$ and all axes of cylinders lie in a plane with polar equations $\theta = 0, \theta = \frac{\pi}{3}$, and $\theta = \frac{2\pi}{3}$. The cross-section is a hexagon, and the shape of the solid looks like the union of two umbrellas.

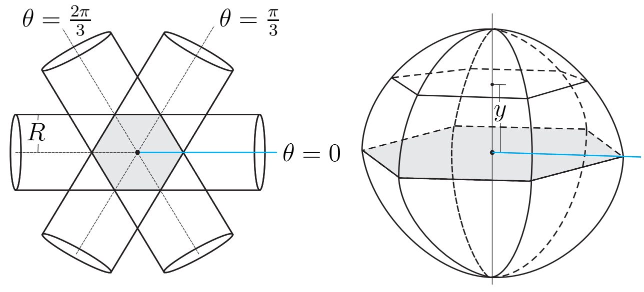


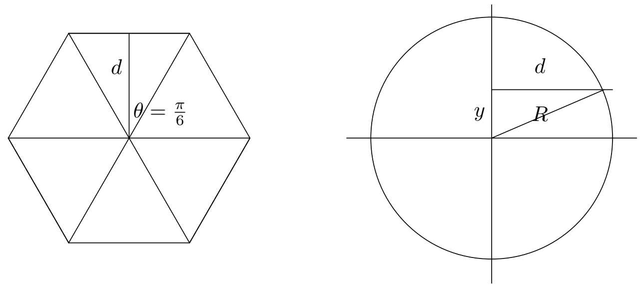

Figure 1: Intersection of three cylinders. (a) Vertical view. (b) The solid.

(a) (6%) Find the area of the cross section of the solid when the height is $y, y \in [-R, R]$.

(b) (6%) Find the volume of the solid.

**Solution:**

(a)Observe the figures below.




When the height is y, the area of the cross section(hexagon) is decided by $d = \sqrt{R^2 - y^2}$.(3%)

Hence,

$$\text{Area} = 6 \times \text{Area of equilateral triangles} = 6 \times \frac{1}{2} \times d \times \frac{2}{\sqrt{3}}d$$
$$= 2\sqrt{3}(R^2 - y^2)\ (3\%)$$

(b) $V = \int_{-R}^{R} A(y)dy(3\%)$

$$= \int_{-R}^{R} 2\sqrt{3}(R^2 - y^2)dy = 2\sqrt{3}(R^2 y - \frac{1}{3}y^3)|_{-R}^{R}(2\%) = \frac{8\sqrt{3}}{3}R^3(1\%)$$

8. (10%) Evaluate the improper integral $\int_0^\infty x^3 e^{-x^2} dx$.

**Solution:**

$$
\begin{aligned}
\int_0^\infty x^3 \mathrm{e}^{-x^2}\,\mathrm{d}x &= \lim_{t\to\infty}\int_0^t x^3\mathrm{e}^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{2}\int_0^t x^2\mathrm{e}^{-x^2}\,\mathrm{d}x^2 = \lim_{t\to\infty}\frac{1}{2}\int_0^{t^2} u\,\mathrm{e}^{-u}\,\mathrm{d}u \ (3\text{分})\\
&= \lim_{t\to\infty} -\frac{1}{2}\int_0^{t^2} u\,\mathrm{d}\mathrm{e}^{-u} = \lim_{t\to\infty} -\frac{1}{2}\left(\left[u\,\mathrm{e}^{-u}\right]\Big|_{u=0}^{u=t^2} - \int_0^{t^2}\mathrm{e}^{-u}\,\mathrm{d}u\right)\ (6\text{分})\\
&= \lim_{t\to\infty}\left(-\frac{1}{2}t^2\,\mathrm{e}^{-t^2} - \frac{1}{2}\left[\mathrm{e}^{-u}\right]\Big|_{u=0}^{u=t^2}\right)\\
&= \lim_{t\to\infty}\left(-\frac{1}{2}t^2\,\mathrm{e}^{-t^2} - \frac{1}{2}\mathrm{e}^{-t^2} + \frac{1}{2}\right)\ (7\text{分}) = \frac{1}{2},
\end{aligned}
$$

where we use the l'Hospital Rule to get find the limit:

$$
\lim_{t\to\infty} t^2\,\mathrm{e}^{-t^2} = \lim_{t\to\infty}\frac{t^2}{\mathrm{e}^{t^2}} \overset{(\frac{\infty}{\infty},L')}{=} \lim_{t\to\infty}\frac{2t}{2t\,\mathrm{e}^{t^2}} = \lim_{t\to\infty}\frac{1}{\mathrm{e}^{t^2}} = 0.
$$

這題重點在於是否具有「瑕積分的觀念」，而判定的依據是以有寫出「極限」為準；縱使前面所有的式子都是寫 $\int_0^\infty$ 或是上、下限寫 $|_0^\infty$， 但最後一定要用極限判斷 $\lim\limits_{t\to\infty} t^2\,\mathrm{e}^{-t^2} = 0$，就認定你知道「瑕積分的意義」；這個極限值不是顯然的，一定要討論，佔 3 分， 也就是說，最後有完整討論這個極限就可以拿到滿分 10 分。 另一個極限式 $\lim\limits_{t\to\infty}\mathrm{e}^{-t^2} = 0$ 可接受直接寫答案，沒有解釋不會扣分。