1031微甲01-04班期末考解答和評分標準

1. (8%) Evaluate $\lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}dt}{x^3}$. (You may use the Mean Value Theorem for Integrals.)

Solution:

Method1.

By the Mean Value Theorem, there is c between x and $\tan x$ such that

$$\frac{\int_x^{\tan x} \sqrt{1+t^3}dx}{\tan x - x} = \sqrt{1+c^3} \quad \textbf{(3pts)}$$

By the Squeeze Theorem,

$$\lim_{x\to 0} x = 0, \quad \lim_{x\to 0} \tan x = 0$$

imply

$$\lim_{x\to 0} c = 0 \quad \textbf{(1pt)}$$

Now we evaluate

$$\begin{aligned}
\lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}dx}{x^3} &= \lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}dx}{\tan x - x} \frac{\tan x - x}{x^3} \\
&= \lim_{x\to 0} \sqrt{1+c^3} \cdot \frac{\tan x - x}{x^3} \\
&= \lim_{x\to 0} \sqrt{1+c^3} \cdot \frac{\sec^2 x - 1}{3x^2} \\
&= \lim_{x\to 0} \sqrt{1+c^3} \cdot \frac{\tan^2 x}{3x^2} \\
&= \lim_{x\to 0} \sqrt{1+c^3} (\frac{\sin x}{x})^2 \frac{1}{3\cos^2 x} \\
&= \sqrt{1+0^3} \cdot 1^2 \cdot \frac{1}{3\cdot 1^2} \\
&= \frac{1}{3} \quad \textbf{(4pt)}
\end{aligned}$$

Method2.

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}dx}{x^3} &= \lim_{x\to 0} \frac{\sqrt{1+\tan^3 x}\sec^2 x - \sqrt{1+x^3}}{3x^2} \quad \textbf{(3pts)} \\
&= \lim_{x\to 0} \frac{\frac{3\tan^2 x \sec^2 x}{2\sqrt{1+\tan^3 x}} \sec^2 x + \sqrt{1+\tan^3 x} \cdot 2\tan x \sec^2 x - \frac{3x^2}{2\sqrt{1+x^3}}}{6x} \quad \textbf{(2pts)} \\
&= \lim_{x\to 0} (\frac{1}{4} \frac{\tan x \sec^4 x}{\sqrt{1+\tan^3 x}} \frac{\tan x}{x} + \frac{1}{3}\sqrt{1+\tan^3 x} \frac{\tan x}{x} \sec^2 x - \frac{x}{4\sqrt{1+x^3}}) \\
&= \frac{1}{3} \quad \textbf{(3pts)}
\end{aligned}$$

Remark.

No point if you are with intent to evaluate $\int \sqrt{1+t^3}dt$.

Calculation error: (**−1pt**) to (**−3pts**) each part.

2. (8%) Let $F(x) = \int_0^x \left(\int_0^{u^3} f(t)dt \right) du$ and $G(x) = \int_0^{x^3} f(u)(x - \sqrt[3]{u})du$, $x \geq 0$. Show that $F(x) = G(x)$ for $x \geq 0$.

Solution:

From

$$\begin{aligned}
\frac{dF(x)}{dx} &= \frac{d}{dx} \int_0^x \left(\int_0^{u^3} f(t)dt \right) du \\
&= \int_0^{x^3} f(t)dt \quad \textbf{(2pts)} \\
\frac{dG(x)}{dx} &= \frac{d}{dx} \int_0^{x^3} f(u)(x - \sqrt[3]{u})du \\
&= \frac{d}{dx} \left(x \int_0^{x^3} f(u)du \right) - \frac{d}{dx} \int_0^{x^3} f(u)\sqrt[3]{u}du \quad \textbf{(2pts)} \\
&= \left(\int_0^{x^3} f(u)du + x \cdot f(x^3) \cdot 3x^2 \right) - f(x^3)\sqrt[3]{x^3} \cdot 3x^2 \\
&= \int_0^{x^3} f(u)du \quad \textbf{(3pts)} \\
\frac{d}{dx}\left(F(x) - G(x)\right) &= \frac{d}{dx}F(x) - \frac{d}{dx}G(x) \\
&= \int_0^{x^3} f(t)dt - \int_0^{x^3} f(u)du = 0
\end{aligned}$$

we have

$$F(x) - G(x) = C$$

for some constant C.

Since

$$\begin{aligned}
F(0) &= \int_0^0 \left(\int_0^{u^3} f(t)dt \right) du = 0 \\
G(0) &= \int_0^{0^3} f(u)(0 - \sqrt[3]{u})du = 0
\end{aligned}$$

we have $C = 0$. **(1pt)**

Therefore, $F(x) = G(x)$.

Remark.

Wrong calculation $\frac{dG(x)}{dx} = \frac{d}{dx} \int_0^{x^3} f(u)(x - \sqrt[3]{u})du = f(x^3)(x - \sqrt[3]{x^3}) = 0$: (**−5pts**).

Calculation error: (**−1pt**) or (**−3pts**) each part.

3. (16%) Three of these six antiderivatives are elementary. Compute them.

(a) $\int x \cos x dx$

(b) $\int \frac{\cos x}{x} dx$

(c) $\int \frac{x}{\ln x} dx$

(d) $\int \frac{\ln(x^2)}{x} dx$

(e) $\int \sqrt{x-1}\sqrt{x}\sqrt{x+1} dx$

(f) $\int \sqrt{x-1}\sqrt{x+1} x dx$

Solution:

(a)
$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + C.$$

(d) Put $u = \ln(x^2)$, $\mathrm{d}u = \frac{2}{x}\,\mathrm{d}x$. So,
$$\int \frac{\ln(x^2)}{x}\,\mathrm{d}x = \int \frac{u}{2}\,\mathrm{d}u = \frac{u^2}{4} + C = \frac{(\ln(x^2))^2}{4} + C.$$

(f) Put $u = x^2 - 1$, $\mathrm{d}u = 2x\,\mathrm{d}x$. So,
$$\int \sqrt{x-1}\sqrt{x+1}x\,\mathrm{d}x = \int x\sqrt{x^2-1}\,\mathrm{d}x = \int \frac{\sqrt{u}}{2}\,\mathrm{d}u = \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2-1)^{\frac{3}{2}}}{3} + C.$$

評分標準:

(1) 僅完成能夠得到答案的變數變換、分部積分或三角代換之步驟，得2分。

(2) 題目(f)使用三角代換但未將$\tan(\sec^{-1}x)$(或其他類似表示式)換成根式，扣2分。

(3) 每次出現少寫積分常數、未將變數代回成x、化簡對數未加絕對值等零星錯誤，但其餘答案皆無誤，扣1分。

(4) 題目(b)、(c)、(e)的任何作答過程與此題計分無關。

4. (16%) (a) Evaluate the integral $\int \frac{dx}{x\sqrt{x^6-1}}$.

(b) Evaluate the integral $\int \frac{(e^{3x}+1)}{(e^{2x}+1)^2}dx$.

Solution:

(a) [6pts] $\int \frac{dx}{x\sqrt{x^6-1}}$

Solution.

$$\begin{aligned} & \int \frac{dx}{x\sqrt{x^6-1}} \\ = & \int \frac{x^5dx}{x^6\sqrt{x^6-1}} && (\text{let} u = \sqrt{x^6-1} du = 3(x^6-1)^{\frac{-1}{2}}x^5dx) \\ = & \frac{1}{3}\int \frac{udu}{(u^2+1)u} \\ = & \frac{1}{3}\int \frac{du}{(u^2+1)} && [+4\text{pts}] \\ = & \frac{1}{3}\tan^{-1}\sqrt{x^6-1} + C && [+2\text{pts}] \end{aligned}$$

note:

$$\begin{aligned} & \int \frac{dx}{x\sqrt{x^6-1}} && (\text{let } x^3 = \sec t 3x^2 = \sec t\tan t dt) \\ = & \int \frac{\sec t\tan t dt}{\sec t|\tan t|} && (\sqrt{x^6-1} = \sqrt{\tan^2 t} = |\tan t|) \quad [+5\text{pts}] \end{aligned}$$

(b) [10pts] $\int \frac{(e^{3x}+1)dx}{(e^{2x}+1)^2}$

Solution.

$$\int \frac{(e^{3x}+1)dx}{(e^{2x}+1)^2} \qquad (\text{let} u = e^x du = e^x dx)$$

$$= \int \frac{(u^3+1)du}{u(u^2+1)^2} \qquad [+2\text{pts}]$$

$$= \int \frac{1}{u} - \frac{u-1}{u^2+1} - \frac{u+1}{(u^2+1)^2} du \qquad [+3\text{pts}]$$

$$= \ln|u| + \tan^{-1} u - \frac{1}{2}\ln(u^2+1) + \frac{1}{2}\frac{1}{u^2+1} - \int \frac{1}{(u^2+1)^2} du \qquad [+2\text{pts}]$$

$$\left(\int \frac{1}{(u^2+1)^2} du = \frac{1}{2}\tan^{-1} e^x + \frac{1}{2}\frac{e^x}{e^{2x}+1} + C\right) \qquad [+3\text{pts}]$$

$$= \ln|e^x| + \tan^{-1} e^x - \frac{1}{2}\ln(e^{2x}+1) + \frac{1}{2}\frac{1}{e^{2x}+1}$$

$$-\frac{1}{2}\tan^{-1} e^x - \frac{1}{2}\frac{e^x}{e^{2x}+1} + C$$

$$= x - \frac{1}{2}\ln(e^{2x}+1) + \frac{1}{2}\tan^{-1} e^x + \frac{1}{2}\frac{1-e^x}{e^{2x}+1} + C$$

不定積分沒有 $+C$ 扣一分

5. (8%) Find the values of s such that $F(s) = \int_0^\infty \sin t\ e^{-st} dt$ converges and evaluate the integrals.

Solution:

"$s \neq 0$" $\int \sin t\ e^{-st} dt = -\frac{\sin t}{s} + \frac{1}{s}\int \cos t\ e^{-st} dt$

$= -\frac{\sin t}{s} - \frac{1}{s^2}\cos t\ e^{-st} - \frac{1}{s^2}\int \sin t\ e^{-st} dt$

$\Longrightarrow \int \sin t\ e^{-st} dt = \frac{-1}{1+s^2}(s \sin t\ e^{-st} + \cos t\ e^{-st})$ (3%)

$\Longrightarrow \lim_{b\to\infty} \int_0^b \sin t\ e^{-st} dt = \frac{-1}{1+s^2} \lim_{b\to\infty} (s \sin b\ e^{-bs} + \cos t\ e^{-bs}) + \frac{1}{1+s^2}$

$\Longrightarrow \int_0^\infty \sin t\ e^{-st} dt = \begin{cases} \frac{1}{1+s^2} & s > 0 \quad (2\%) \\ diverge & s < 0 \quad (1\%) \end{cases}$

"s=0" $F(0) = \int_0^\infty \sin t\ dt = \lim_{b\to\infty}(-\cos b + 1)$ $\quad$ *diverges* $\quad$ (2%)

6. (8%) Find the length of the loop of the curve $3ay^2 = x(a-x)^2$, $a > 0$.

Solution:

$3ay^2 = x(a-x)^2, a > 0$

$\Longrightarrow 6ayy' = (a-x)^2 - 2x(a-x) = (a-x)(a-3x)$

$\Longrightarrow y' = \frac{(a-x)(a-3x)}{6ay} = \frac{(a-3x)}{\sqrt{12ax}}$

Length $= 2\int_0^a \sqrt{1 + (\frac{dy}{dx})^2} dx = 2\int_0^a \sqrt{1 + \frac{(a-3x)^2}{12ax}} dx$ (2%)

$= 2\int_0^a \frac{a+3x}{\sqrt{12ax}} dx = \frac{1}{\sqrt{3a}} \int_0^a (ax^{-1/2} + 3x^{1/2})\ dx$ (3%)

$= \frac{4}{\sqrt{3}} a$ (3%)

7. (20%) Let C be the curve $y = \ln x$, $0 < x \leq 1$ and R be the region bounded by x-axis, y-axis and the curve C.
   (a) Compute the area of R if it is finite.
   (b) Compute the arc length of C if it is finite.
   (c) Rotate C about the y-axis. Compute the area of the generated surface if it is finite.
   (d) Rotate R about the line $y = x$. Compute the volume of the generated solid if it is finite.

Solution:

(5%)(a) Area(R) $= -\int_0^1 \ln x \, dx$ (3%)

$= -\lim_{t\to 0^+} (x\ln x - x)|_t^1$

$= 1 + \lim_{t\to 0^+} t\ln t$ (1%)

$= 1$ (1%)

(5%)(b) Length(C) $= \int_0^1 \sqrt{1+(y')^2} \, dx$ (3%)

$= \int_0^1 \sqrt{1+(\frac{1}{x})^2} \, dx$

$\geq \int_0^1 \frac{1}{x} \, dx$ (1%)

$= \infty$ (1%)

(5%)(c) $S = 2\pi \int_0^1 x \, ds$

$= 2\pi \int_0^1 x\sqrt{1+(\frac{1}{x})^2} \, dx$ let $x = \tan\theta, dx = \sec^2\theta d\theta$ (2%)

$= 2\pi \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$ (1%)

$= \frac{2\pi}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\Big|_0^{\frac{\pi}{4}}$

$= \pi(\sqrt{2} + \ln(\sqrt{2}+1))$ (2%)

(5%)(d) $\bar{x} = -\int_0^1 x\ln x \, dx/\text{Area}(R)$

$= -\big(\lim_{t\to 0^+} (\frac{x^2}{2}\ln x)|_t^1 - \int_0^1 \frac{x}{2} \, dx\big)/1$

$= \frac{1}{4}$ (2%)

$\bar{y} = -\frac{1}{2}\int_0^1 (\ln x)^2 \, dx/\text{Area}(R)$

$= -\frac{1}{2}\big(\lim_{t\to 0^+} (x(\ln x)^2)|_t^1 - 2\int_0^1 \ln x \, dx\big)/1$

$= -1$ (2%)

The distance from $(\bar{x}, \bar{y})$ to the line $y = x$ is $\frac{5}{4\sqrt{2}}$

By Puppus Theorem, volume $= 2\pi\frac{5}{4\sqrt{2}}\text{Area}(R) = \frac{5\sqrt{2}}{4}\pi$ (1%)

8. (8%) (a) Sketch the curve with polar equation $r = 1 + 2\cos 2\theta$.

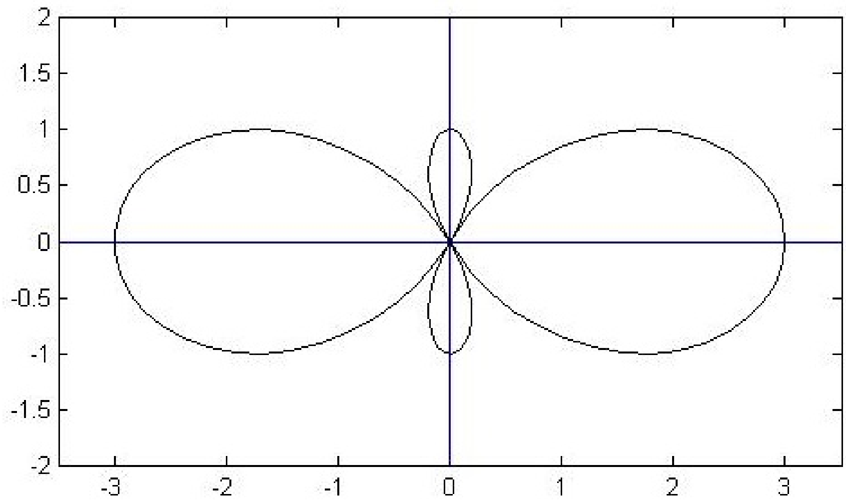
(b) Find the area of the region inside both curves $r = 1$ and $r = 1 + 2\cos 2\theta$.

Solution:

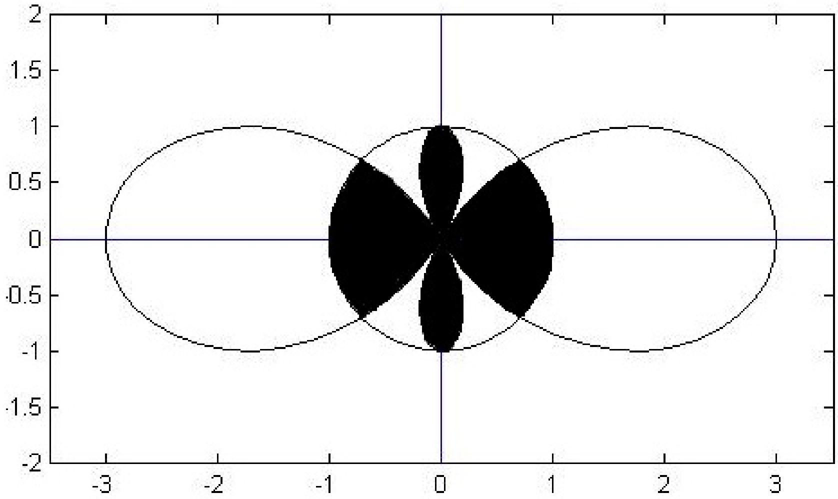
(a) First find some values of r for different θ :

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	π
r	3	1	0	-1	0	1	3

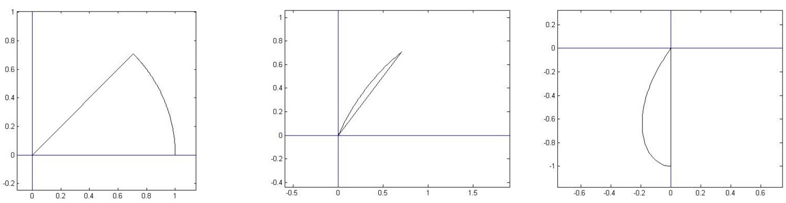
Since the graph is symmetric about the x-axis, so it looks like:(2 pts)



(b) The region to be integrated is as follows :(1pt)



Divide the required region into 3 parts:



(a) I (b) II (c) III

The areas of these three parts are represented by the following integrals: (1pt for each)

I. $1^2 \cdot \pi \cdot \frac{45°}{360°} = \frac{\pi}{8}$

II. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}(1 + 2\cos 2\theta)^2 d\theta$

III. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(1 + 2\cos 2\theta)^2 d\theta = \frac{1}{2}(3\theta + 2\sin 2\theta + \frac{1}{2}\sin 4\theta\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8} - 1$

Hence the area is 4(I + II + III)

$= \frac{\pi}{2} + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + 2\cos 2\theta)^2 d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta)^2 d\theta$

$= \frac{\pi}{2} + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta)^2 d\theta$

$= 2(3\theta + 2\sin 2\theta + \frac{1}{2}\sin 4\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}) = 2(\frac{3}{2}\pi - \frac{3}{4}\pi - 2) + \frac{\pi}{2} = 2\pi - 4.$ (2pts)

9. (10%) An inverted cycloid is defined by the parametried equations

$$x(\theta) = r(\theta - \sin\theta),\ y(\theta) = -r(1-\cos\theta),\ 0 \le \theta \le 2\pi.$$

Consider the motion of a particle without friction and rolling down the inverted cycloid released from $(x(\alpha), y(\alpha))$, where $\alpha \in [0,\pi]$. By the conservation of energy, the velocity of the particle at $(x(\theta), y(\theta))$ is given by

$$v = \frac{ds}{dt} = \sqrt{2gr(\cos\alpha - \cos\theta)} \qquad (*)$$

where s is the arc length function and $\alpha \le \theta \le \pi$.

(a) Derive a separable differential equation for $\frac{d\theta}{dt}$ from (*).

(b) Compute the time $T = \int_{\theta=\alpha}^{\pi} dt$ for the particle to get to the lowest point $(x(\pi), y(\pi))$.

Solution:

(a) (4 points)

We found $\frac{dx}{d\phi}$ and $\frac{dy}{d\phi}$ first:

$\frac{dx}{d\phi}$ was $r - r\cos\phi$, and $\frac{dy}{d\phi}$ was $-r\sin\phi$.

$S(\theta) = \int_\alpha^\theta \sqrt{(\frac{dx}{d\phi})^2 + (\frac{dy}{d\phi})^2}\ d\phi$

$= \sqrt{2}r\int_\alpha^\theta \sqrt{1-\cos\phi}\ d\phi.$

$v = \frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \frac{ds}{dt}\frac{d\theta}{ds}$

Therefore, $\frac{d\theta}{dt} = \sqrt{\frac{g(\cos\alpha - \cos\theta)}{r(1-\cos\theta)}}$

(b) (6 ponits)

$T = \int_{\theta=\alpha}^{\pi} dt$

$= \int_{\theta=\alpha}^{\pi} \frac{dt}{d\theta}\ d\theta$

$= \int_{\theta=\alpha}^{\pi} \sqrt{\frac{r(1-\cos\theta)}{g(\cos\alpha - \cos\theta)}}\ d\theta$

Consider $\int \sqrt{\frac{r(1-\cos\theta)}{g(\cos\alpha - \cos\theta)}}\ d\theta$,

Let $A = \cos\alpha$, $u = \cos\frac{\theta}{2}$ and $du = \frac{-1}{2}\sin\frac{\theta}{2}\ d\theta$

The above equation $= -2\sqrt{2}\int \frac{1}{\sqrt{A+1-2u^2}}du$

Then, Let $\sin t = \sqrt{\frac{2}{A+1}}u$, and $\cos t dt = \sqrt{\frac{2}{A+1}}du$.

The above equation $= \frac{-2\sqrt{2}}{\sqrt{A+1}}\sqrt{\frac{A+1}{2}}\int \frac{\cos t}{\cos t}dt$

$= -2t$

$= -2\sin^{-1}(\frac{1}{\cos\frac{\alpha}{2}} \cdot \cos\frac{\theta}{2})$

Consequently,

$T = \sqrt{\frac{r}{g}}(-2\sin^{-1}(\frac{1}{\cos\frac{\alpha}{2}} \cdot \cos\frac{\alpha}{2}))\ |_\alpha^\pi$

$= \pi\sqrt{\frac{r}{g}}.$

Other proper methods are permitted to solve these problems.

10. (8%) Solve the differential equation $y' + \frac{2}{x}y = \frac{y^3}{x^2}$. (Hint: let $y^2 = \frac{1}{u}$.)

Solution:

$y' + P(x)y = Q(x)y^n$, where $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x^2}$ and $n = 3$.
We found that this equation is Bernoulli equation.

Consequently, we mutiplied y^{-3} at both sides.
The original equation $\Rightarrow y^{-3}y' + y^{-3}(\frac{2}{x}y) = \frac{1}{x^2}$
$\Rightarrow y^{-3}y' + \frac{2}{x}y^{-2} = \frac{1}{x^2}$.
Let $y^2 = \frac{1}{u}$, that is, $u = y^{-2}$.
After that, we calculated and found that $u' = -2y^{-3}y'$.
Therefore, the above equation $\Rightarrow -u' + \frac{4}{x}u = \frac{2}{x^2}$. (3 ponits)
The integral factor: $I(x) = e^{\int \frac{4}{x}udx} = e^{-4\ln x} = x^{-4}$. (2 points)

We mutiplied x^{-4} on both sides of above equation, and we got: $-x^{-4}u' + 4x^{-5}u = 2x^{-6}$.
$\Rightarrow \frac{d}{dx}(-x^{-4}u) = 2x^{-6}$
We integrated both sides, and we had: $-x^{-4}u = \int 2x^{-6}dx + C$, where C is a constant.
$\Rightarrow -x^{-4}u = \frac{2}{5}x^{-5} + C$, where C is a constant.
$\Rightarrow -u = \frac{2}{5x} + Cx^4$.
We known that $u = \frac{1}{y^2}$.
Terefore, the solution of this problem is that $y^{-2} = \frac{2}{5x} + Cx^4$, where C is a constant.
$\Rightarrow y^2 = \frac{1}{\frac{2}{5x} + cx^4}$, where C is a constant. (3 points)
Other proper methods are permitted to solve this problem.