1022微甲07-11班期中考解答和評分標準

1. (10%) Find the values of ρ for the convergence of the series below

(a) $\displaystyle\sum_{n=0}^{\infty} e^{n(\rho^2-\rho-2)}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}-1}{n^{\rho}}$.

Solution:

(a) This geometric series converges *if and only if* the ratio

$$e^{\rho^2-\rho-2} < 1, \quad \text{or} \quad \rho^2-\rho-2<0 \qquad \text{(3 points)}$$

$$(\rho-2)(\rho+1)<0 \quad \therefore\ -1<\rho<2 \qquad \text{(2 points)}$$

(b) Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

$$e^{\frac{1}{n}}-1 \approx \frac{1}{n} \qquad \text{(3 points)}$$

Apply the limit comparison test to $\displaystyle\sum \frac{e^{\frac{1}{n}}-1}{n^{\rho}}$ and $\displaystyle\sum \frac{1}{n^{\rho+1}}$

$$\lim_{n\to\infty} \frac{\frac{e^{\frac{1}{n}}-1}{n^{\rho}}}{\frac{1}{n^{\rho+1}}} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim_{\frac{1}{n}=t\to 0^+} \frac{e^t-1}{t} = 1 \in (0,\infty)$$

Thus two series both converges or diverges. (1 point)

$\displaystyle\sum \frac{1}{n^{\rho+1}}$ is a p-series, which converges *if and only if* $1+\rho=p>1$

$$\therefore \sum \frac{e^{\frac{1}{n}}-1}{n^{\rho}} \text{ converges } \textit{if and only if } \rho>0 \qquad \text{(1 point)}$$

2. (10%) (a) Prove $\ln(n+1) < 1+\frac{1}{2}+\cdots+\frac{1}{n} < 1+\ln n$.

(b) Test for convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$.

Solution:

(a) $f(x)=\dfrac{1}{x}, g(x)=\dfrac{1}{1+x}$

$\displaystyle\int_0^n g(x)dx \leqslant 1+\frac{1}{2}+\cdots+\frac{1}{n} \leqslant f(1)+\int_1^n f(x)dx$

you can graph the picture.

(b) by (a)

$a_n \geqslant \dfrac{1}{1+lnn} \geqslant \dfrac{1}{1+n}$

By comparison test $\sum a_n$ div.

3. (15%) Let $f(x,y)=\sin(x-y)e^{-x^2-y^2}$, $P=(\sqrt{2},\sqrt{2})$.

(a) Find the maximum rate of change of f at P

(b) Find the direction in which the maximum rate of change occurs.

(c) Find the directional derivative $D_{\mathbf{u}}(P)$, where $\mathbf{u}=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

Solution:

(a) 9 points

$\nabla f(x,y) = <\cos(x-y)e^{-x^2-y^2} - 2x\sin(x-y)e^{-x^2-y^2}, -\cos(x-y)e^{-x^2-y^2} - 2y\sin(x-y)e^{-x^2-y^2}>$

(f_x 2 points,f_y 2 points)

$\nabla f(\sqrt{2},\sqrt{2}) = <e^{-4}, -e^{-4}>$

(e^{-4} 1 point,$-e^{-4}$ 1 point)

the maximum rate of change is $|\nabla f(\sqrt{2},\sqrt{2})| = \dfrac{\sqrt{2}}{e^4}$

(formula 2 points,answer 1 point)

(b) 3 points

the direction is the same as $\nabla f(\sqrt{2},\sqrt{2})$,which is the direction of $<e^{-4}, -e^{-4}>$

(any vector that is the same direction as $<1,-1>$ will be valid and get 3 points.But if $|\nabla f(\sqrt{2},\sqrt{2})|$ is wrong in (a),one can get only 0 point! If one just claim "*the direction is parallel to* $\nabla f(\sqrt{2},\sqrt{2})$" but no write down "$<e^{-4}, -e^{-4}>$" explicitly,he can get only 2 points because opposite directions are included and they cause minimum not maximum.)

(c) 3 points

$D_u f(P) = \nabla f(P) \cdot u = \dfrac{1-\sqrt{3}}{2e^4}$

(formula 2 points,answer 1 point)

4. (15%) Let $\mathbf{r}(t) = \left\langle \frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}} \right\rangle$ for $t \geq 0$.

(a) Find the length of the arc $0 \leq t \leq 2$ of $\mathbf{r}(t)$.

(b) Find the curvature $\kappa(t)$.

(c) Find $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$, the principal unit normal vector and the binormal unit vector when $t = 1$ respectively.

Solution:

$\overrightarrow{r(t)} = (\frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}})$

(a)

$\overrightarrow{r'(t)} = (2t^3, 1, 2t^{\frac{3}{2}})$. Then $|\overrightarrow{r'(t)}| = 1 + 2t^3$. Therefore, $L = \int_0^2 (1+2t^3) = 10$ [3pts]

(b)

$\overrightarrow{r''(t)} = (6t^2, 0, 3t^{\frac{1}{2}})$. Then $\overrightarrow{r'(t)} \times \overrightarrow{r''(t)} = (3t^{\frac{1}{2}}, 6t^{\frac{7}{2}}, -6t^2)$ and $|\overrightarrow{r'(t)} \times \overrightarrow{r''(t)}| = 3t^{\frac{1}{2}}(2t^3+1)$

Therefore, the curvature $k(t = \dfrac{|\overrightarrow{r'(t)} \times \overrightarrow{r''(t)}|}{|\overrightarrow{r'(t)}|^3} = \dfrac{3\sqrt{t}}{(2t^3+1)^2}$[3pts]

(c)

$\overrightarrow{T(t)} = \dfrac{\overrightarrow{r'(t)}}{|\overrightarrow{r'(t)}|} = \dfrac{1}{1+2t^3}(2t^3, 1, 2t^{\frac{3}{2}}) \Rightarrow \overrightarrow{T(1)} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ [3pts]

$\overrightarrow{T'(t)} = \dfrac{1}{(1+2t^3)^2}(6t^2, -6t^2, -6t^{\frac{7}{2}} + 3t^{\frac{1}{2}}) \Rightarrow \overrightarrow{T'(1)}) = (\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

Therefore, $\overrightarrow{N(1)} = \dfrac{\overrightarrow{T'(1)}}{\mid \overrightarrow{T'(1)} \mid} = (\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$ [3pts]

$\overrightarrow{B(1)} = \overrightarrow{T(1)} \times \overrightarrow{N(1)} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$ [3pts]

5. (15%) Find the extreme values of $f(x,y) = x^2y - xy + xy^2$ on $x^2 + xy + y^2 - x - y = 1$.

Solution:
Define $g(x,y) = x^2 + xy + y^2 - x - y$
Use the method of Lagrange multipliers:

$$\begin{cases} \nabla f & = & \lambda \nabla g \\ g & = & 1 \end{cases}$$

$$\Rightarrow \begin{cases} \partial/\partial x: & y(2x+y-1) & = & \lambda(2x+y-1) & \cdots(1) \\ \partial/\partial y: & x(x+2y-1) & = & \lambda(x+2y-1) & \cdots(2) \\ & x^2+xy+y^2-x-y & = & 1 & \cdots(3) \end{cases}$$

(6% up to this point.)

First we note that when $\nabla g = \overrightarrow{0}$, $(x,y) = (1/3, 1/3)$ is not on $g(x,y) = 1$.

Case 1: $2x + y - 1 = 0$
$\Rightarrow y = 1 - 2x$, λ: any.
Substitute into (3) we have $(x,y) = (\frac{-1}{3}, \frac{5}{3})$ or $(1, -1)$

Case 2: $2x + y - 1 \neq 0 \Rightarrow \lambda = y$
(I) $x + 2y - 1 = 0 \Rightarrow x = 1 - 2y$
Substitute into (3) we have $(x,y) = (\frac{5}{3}, \frac{-1}{3})$ or $(-1, 1)$

(II) $x + 2y - 1 \neq 0 \Rightarrow \lambda = x = y$
Substitute into (3) we have $(x,y) = (\frac{-1}{3}, \frac{-1}{3})$ or $(1, 1)$
(6% for the above discussions.)

Substitute these points into f we find that

$f(\frac{-1}{3}, \frac{5}{3}) =$	$f(\frac{5}{3}, \frac{-1}{3}) =$	$f(\frac{-1}{3}, \frac{-1}{3}) = \frac{-5}{27}$	are the minima, while
$f(1, -1) =$	$f(-1, 1) =$	$f(1, 1) = 1$	are the maxima.

(3% for the conclusion.)

6. (10%) Find the local maximum, and local minimum values and saddle point(s) of $f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 3$.

Solution:
$f_x = 6xy - 6x = 0, 6x(y-1) = 0, x = 0$ or $y = 1$
$f_y = 3y^2 + 3x^2 - 6y = 0, 3x^2 + (3y^2 - 6y + 3) = 3, x^2 + (y-1)^2 = 1$

if $x = 0, y = 0, 2$
if $y = 1, x = -1, 1$

We have 4 critical points: $(0,0), (0,2), (1,1), (-1,1)$

$f_{xx} = 6y - 6$
$f_{xy} = f_{yx} = 6x$
$f_{yy} = 6y - 6$

$D(x,y) = (6y-6)^2 - (6x)^2 = 36[(y-1)^2 - x^2]$

$D(0,0) = 36 > 0, f_{xx}(0,0) = -6 < 0$, local maximum. $f(0,0) = 3$
$D(0,2) = 36 > 0, f_{xx}(0,2) = 6 > 0$, local minimum. $f(0,2) = -1$
$D(1,1) = -36 < 0$
$D(-1,1) = -36 < 0$

Ans: local maximum $f(0,0) = 3$
local minimum $f(0,2) = -1$
Two saddle points at $(1,1)$ and $(-1,1)$

Grading Policy:

f_x and f_y: 2 points
4 critical points: 2 points
$D(x,y)$: 2 points
local maximum/minimum values: 2 points
2 saddle points: 2 points

7. (10%) (a) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$.

(b) Let $f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$ when the power series is convergent. Evaluate $f^{(3)}(1)$.

Solution:

(a) By the ratio test, the series converges when
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\sqrt{n}}{\sqrt{n+1}}||\frac{x-1}{-2}| = |\frac{x-1}{-2}| < 1 \Rightarrow |x-1| < 2 = R$ (3%)
Check the endpoints:
(1%)(1)$x = 3, \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by alternating series test(1%).
(1%)(2)$x = -1, \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series with $p = \frac{1}{2} < 1$.
By(1)&(2), the interval of convergence $I = (-1, 3]$.

(b) Since $f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$ (3%)
Hence, $f^{(3)}(1) = \frac{3!}{-8\sqrt{3}} = -\frac{\sqrt{3}}{4}$ (1%)
If you compute $f^{'}(x)$, $f^{''}(x)$, and $f^{'''}(x)$. Then you get zero point if the answer is not correct.

8. (15%) (a) Write down the general terms the MacLaurin series of $\sin x$ and $\sin^{-1} x$.

(b) Find their radii of convergence.

(c) Find $\lim_{x\to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$.

Solution:

(a)(b)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{..........} (4\ pts)$$

radius of convergence $= \infty$..........(1 *pt*)

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{k} (-t^2)^k dt$$

$$= \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{k} (-1)^k \int_0^x t^{2k} dt = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{k} \frac{(-1)^k}{2k+1} x^{2k+1} \text{..........} (5\ pts)$$

, where $\binom{\frac{-1}{2}}{k} = \frac{(\frac{-1}{2})(\frac{-1}{2}-1)\cdots(\frac{-1}{2}-k+1)}{k!}$

radius of convergence $= 1$..........(1 *pt*)

(c)

We first compute the power series of the product

$$\sin x \sin^{-1} x = [x - \frac{x^3}{6} + \frac{1}{120}x^5 + \cdots] \cdot [x + (\frac{-1}{2})(\frac{-1}{3})x^3 + \frac{(\frac{-1}{2})(\frac{-1}{2}-1)}{2!}\frac{x^5}{5!} + \cdots]$$

$$= [x - \frac{x^3}{6} + \frac{1}{120}x^5 + \cdots] \cdot [x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots]$$

$$= (x^2 + \frac{1}{18}x^6 + \frac{1}{30}x^8 + \cdots) \text{..........} (2\ pts)$$

So

$$\lim_{x\to 0} \frac{\sin x \sin^{-1} x - x^2}{x^6} = \lim_{x\to 0} \frac{(x^2 + \frac{1}{18}x^6 + \frac{1}{30}x^8 + \cdots) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{\frac{1}{18}x^6 + \frac{1}{30}x^8 + \cdots}{x^6} = \lim_{x\to 0} (\frac{1}{18} + \frac{1}{30}x^2 + \cdots) = \frac{1}{18} \text{..........} (2\ pts)$$