1022微甲07-11班期末考解答

1. (10%) Evaluate $\iint\limits_A e^{xy}\,dxdy$, where A is the region enclosed by $xy = 1$, $xy = 4$, $y = 1$ and $y = 3$.

Solution:

<**Solution-1**>

$\iint_A e^{xy}dxdy = \int_1^3 \int_{1/y}^{4/y} e^{xy}dxdy = \int_1^3 \frac{1}{y}e^{xy}|_{1/y}^{4/y}dy = (e^4 - e)\int_1^3 \frac{1}{y}dy$ (5%)

$= (e^4 - e)\ln y|_1^3 = \ln 3(e^4 - e)$ (5%)

<**Solution-2**>

let $u = xy$, $v = y$

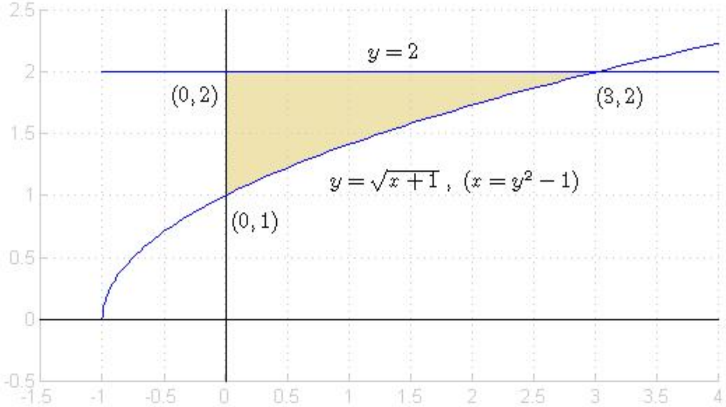
$\iint_A e^{xy}dxdy = \iint_A e^u|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}|dudv$

$= \int_1^3 \int_1^4 e^u \frac{1}{v}dudv(2\%) = (e^4 - e)\int_1^3 \frac{1}{v}dv$ (3%)

$= (e^4 - e)\ln y|_1^3 = \ln 3(e^4 - e)$ (5%)

2. (10%) Sketch the region of integration and evaluate the integral $\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}}\, dy dx$.

Solution:



(4%)

$$
\begin{aligned}
\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}}\, dy\, dx &= \int_1^2 \int_0^{y^2-1} e^{\frac{x}{y+1}}\, dx\, dy \quad (3\%) \\
&= \int_1^2 (y+1) e^{\frac{x}{y+1}} \Big|_{x=0}^{y^2-1} dy \\
&= \int_1^2 (y+1)\big(e^{y-1} - 1\big)\, dy \\
&= \int_1^2 y e^{y-1}\, dy + \int_1^2 e^{y-1}\, dy - \int_1^2 (y+1)\, dy \\
&= y e^{y-1}\Big|_1^2 - \int_1^2 e^{y-1}\, dy + \int_1^2 e^{y-1}\, dy - \frac{1}{2}(y+1)^2\Big|_1^2 dy \\
&= 2e - 1 - \frac{5}{2} = 2e - \frac{7}{2} \quad (3\%)
\end{aligned}
$$

3. (15%) Let D be the bounded region in the first quadrant enclosed by $y = 0$, $x = 1$, and $y = \sqrt{x}$ with positively oriented boundary C (i.e. counter clockwise.). Evaluate

$$\oint_C \left[9x^2y(x^3+1)^{\frac{1}{2}} - xy^2(x^3+1)^{\frac{3}{2}}\right] dx + \left[2(x^3+1)^{\frac{3}{2}} + 2(y^3+1)^{\frac{3}{2}}\right] dy.$$

Solution:

Let D be the bounded region in the first quadrant enclosed by $y = 0, x = 0$ and $y = \sqrt{x}$ with positively oriented boundary C. Evaluate

$$\oint_C \left[9x^2y\left(x^3+1\right)^{1/2} - xy^2\left(x^3+1\right)^{3/2}\right] dx + \left[2\left(x^3+1\right)^{3/2} + 2\left(y^3+1\right)^{3/2}\right] dy.$$

Proof: Let $P(x,y) = 9x^2y\left(x^3+1\right)^{1/2} - xy^2\left(x^3+1\right)^{3/2}$ and $Q(x,y) = 2\left(x^3+1\right)^{3/2} + 2\left(y^3+1\right)^{3/2}$. We have

$$\begin{aligned} \frac{\partial Q}{\partial x} &= 9x^2\left(x^3+1\right)^{1/2}.(3\ \%) \\ \frac{\partial P}{\partial y} &= 9x^2\left(x^3+1\right)^{1/2} - 2xy\left(x^3+1\right)^{3/2}.(3\ \%) \end{aligned}$$

By Green's theorem, we have

$$\begin{aligned} &\oint_C \left[9x^2y\left(x^3+1\right)^{1/2} - xy^2\left(x^3+1\right)^{3/2}\right] dx + \left[2\left(x^3+1\right)^{3/2} + 2\left(y^3+1\right)^{3/2}\right] dy \\ =& \oint_C Pdx + Qdy \\ =& \int\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ =& \int_0^1 \int_0^{\sqrt{x}} 2xy\left(x^3+1\right)^{3/2} dydx\ (5\ \%) \\ =& \int_0^1 x^2\left(x^3+1\right)^{3/2} dx \\ =& \frac{2}{15}\left(x^3+1\right)^{5/2}|_{x=0}^{x=1} \\ =& \frac{2}{15}\left(2^{5/2}-1\right).(4\ \%) \end{aligned}$$

4. (10%) Evaluate the triple integral $\iiint_E xyz\,dV$ with

$$E = \left\{0 \le x \le \sqrt{4-y^2},\ 0 \le y \le 2,\ \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2}\right\}.$$

Solution:

Evaluate the triple integral $\iiint_E xyzdV$

$E = \{0 \le x \le \sqrt{4-y^2}, 0 \le y \le 2, \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2}\}$

There are three method to do it.

(1)

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xyzdzdxdy \text{ (4pt)}$$

$$= \int_0^2 \int_0^{\sqrt{4-y^2}} 4xy - x^3y - xy^3 dxdy \text{(2pt)}$$

$$= \int_0^2 2y(4-y^2) - y\frac{(4-y^2)^2}{4} - y^3\frac{4-y^2}{2} dy \text{(2pt)}$$

$$= \frac{8}{3} \text{(2pt)}$$

(2)use cylindrical coordinates

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_r^{\sqrt{8-r^2}} r^2 \sin\theta\cos\theta zrdzdrd\theta \text{(6pt)}$$

$$= \int_0^{\frac{\pi}{2}} r^3 \sin\theta\cos\theta(4-r^2)drd\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\sin\theta\cos\theta - \frac{32}{3}\sin\theta\cos\theta d\theta$$

$$= \frac{8}{3}$$

(3)use spherical coordinates

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} \rho^3 \sin^2\phi\sin\theta\cos\theta\cos\phi\rho^2\sin\phi d\rho d\theta d\phi \text{(6pt)}$$

$$= \frac{8\cdot 8\cdot 8}{6} \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta \cdot \int_0^{\frac{\pi}{4}} \sin^3\phi d\sin\phi$$

$$= \frac{8}{3}$$

In methods (2) and (3),if you get first 6 points you may get partial credit depend on how much you complete.

5. (10%) Find the area of the surface $\{x^2+y^2+z^2=4,\ 1\le x^2+y^2\le 3,\ z\ge 0\}$.

Solution:

(method 1)

$A(S)=\iint_S 1dS=\iint_{1\le x^2+y^2\le 3}\sqrt{1+z_x^2+z_y^2}dA=\int_0^{2\pi}\int_1^{\sqrt{3}}\frac{2r}{\sqrt{4-r^2}}drd\theta$

$=2\pi\cdot(-2)\sqrt{4-r^2}|_1^{\sqrt{3}}=4\pi(\sqrt{3}-1).$

(method 2)

$A(S)=\int_{\pi/6}^{\pi/3}\int_0^{2\pi}2^2\sin\phi d\theta d\phi=-8\pi\cos\phi|_{\pi/6}^{\pi/3}=4\pi(\sqrt{3}-1).$

(method 3)

$A(S)=\int_{\phi=\pi/6}^{\phi=\pi/3}2\pi\cdot 2\sin\phi ds=\int_{\phi=\pi/6}^{\phi=\pi/3}2\pi\cdot 2\sin\phi 2d\phi=4\pi(\sqrt{3}-1).$

6. (15%) Let S be the part of the sphere $x^2 + y^2 + (z-2)^2 = 8$ that lies above the xy-plane and that has outward normal (i.e. with $\mathbf{k}$-component ≥ 0). Let $\mathbf{F}(x,y,z) = \langle -y^3 \cos xz,\ x^3 e^{yz},\ -e^{xyz} \rangle$. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Solution:

Method 1. Let C be the boundary of S, which is $x^2 + y^2 = 4$, $z = 0$, with counterclockwise orientation. One possible parametrization is
$\mathbf{r}(t) =< 2\cos t, 2\sin t, 0 >, 0 \leq t \leq 2\pi$, and $\mathbf{r}'(t) =< -2\sin t, 2\cos t, 0 >$. .... 4 points

By Stokes' Theorem $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$. .... 4 points

Since $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = 16 \int_0^{2\pi} (\sin^4 t + \cos^4 t)\, dt$, ... 4 points

one finally has $\oint_C \mathbf{F} \cdot d\mathbf{r} = 16 \left(\frac{3}{4} t + \frac{\sin 4t}{16} \right) \Big|_0^{2\pi} = 12 \cdot 2\pi = 24\pi$. .... 3 points

Method 2. Let C be the positive boundary of S and $D = \{x^2 + y^2 \leq 4, z = 0\}$ oriented with unit normal $\mathbf{k} = (0,0,1)$. Note that C is also the positive boundary of D. .... 4 points

Apply Stokes' Theorem twice $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$. .... 4 points

Since D is on xy-plane and $\operatorname{curl}\mathbf{F} \cdot \mathbf{k} = 3x^2 + 3y^2$ on $D = \{x^2 + y^2 \leq 4, z = 0\}$,
$\iint_D \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \operatorname{curl}\mathbf{F} \cdot \mathbf{k}\, dS = 3 \iint_D x^2 + y^2\, dA$. .... 4 points

Using polar coordinates, the answer is $\int_0^{2\pi} \int_0^2 3r^2 \cdot r\, dr d\theta = 2\pi \cdot \frac{3r^4}{4} \Big|_0^2 = 24\pi$. .... 3 points

7. (15%) (a) Find a scalar function $f(x,y,z)$ such that $\nabla f = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$.

(b) Find the line integral $\int_C \sin y\,dx + x\cos y\,dy + (y - \sin z)dz$, where $C:\ \mathbf{r}(t) = \left\langle t, \frac{\pi}{2}\cos t, \frac{\pi}{2}\sin t\right\rangle$, $0 \le t \le \pi$.

Solution:

(a)

$$\nabla f = \sin y\vec{i} + x\cos y\vec{j} - \sin z\vec{k}$$

Let $f = \int \sin y\ dx = x\sin y + h(y,z)$ (3%)

$$\begin{aligned}
\frac{\partial f}{\partial y} &= x\cos y + \frac{\partial h}{\partial y} = x\cos y \\
&\Rightarrow \frac{\partial h}{\partial y} = 0 \\
&\Rightarrow h(y,z) = g(z) \quad \text{for some} \quad g
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial z} &= g'(z) = -\sin z \\
&\Rightarrow g(z) = \cos z + C
\end{aligned}$$

Thus $f = x\sin y + \cos z + C$, C is a constant (3%)

(b)

$$\begin{aligned}
\vec{r}(t) &= < t, \frac{\pi}{2}\cos t, \frac{\pi}{2}\sin t > \\
\vec{r}(0) &= < 0, \frac{\pi}{2}, 0 > \\
\vec{r}(\pi) &= < \pi, -\frac{\pi}{2}, 0 >
\end{aligned}$$

$$\begin{aligned}
& \int_C \sin y dx + x\cos y dy + (y - \sin z)dz \\
=\ & \int_C \sin y dx + x\cos y dy - \sin z dz + \int_C y dz \\
=\ & f(\pi, -\frac{\pi}{2}, 0) - f(0, \frac{\pi}{2}, 0) + \int_0^\pi (\frac{\pi}{2}\cos t)(\frac{\pi}{2}\cos t)dt \quad (3\%) \\
=\ & -\pi + 1 - 1 + (\frac{\pi}{2})^2\,(\frac{t + \frac{1}{2}\sin 2t}{2})\Big|_0^\pi \\
=\ & -\pi + (\frac{\pi}{2})^3 \quad (3\%)
\end{aligned}$$

其他評分標準

1. 少寫常數C，扣一分
2. $\int_C y dz = \frac{\pi^3}{8}$ (3%)

8. (15%) Let $\mathbf{F} = \left\langle 3xy^2,\, y^3,\, e^{x^2+y^2} \right\rangle$. Let S be the part of the surface $z = 1 - x^2 - y^2$ that lies above xy-plane oriented upwards (that is, with normal having $\mathbf{k}$-component ≥ 0). Calculate the flux $\int_S \mathbf{F} \cdot d\mathbf{S}$ of $\mathbf{F}$ across S. Note that S is not closed.

Solution:

Method 1. Direct computation Let $D = \{(x,y) | x^2 + y^2 \leq 1\}$, and a possible parametrization of S is $\mathbf{r}(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle$, $\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle$. .... 3 points

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(x,y)) \cdot \mathbf{r}_x \times \mathbf{r}_y \, dA = \iint_D 6x^2y^2 + 2y^4 + e^{x^2+y^2} \, dA$. .... 5 points

Using polar coordinates $\iint_D 6x^2y^2 \, dA = \pi/4$, ... 2 points

$\iint_D 2y^4 \, dA = \pi/4$, ... 2 points

$\iint_D e^{x^2+y^2} \, dA = \pi(e-1)$, ... 3 points

and the total flux is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \pi(e - \frac{1}{2})$.

Method 2. Applying Divergence Theorem Let $D = \{(x,y) | x^2+y^2 \leq 1\}$, $S_1 = \{(x,y,z) | (x,y) \in D,\ z = 0\}$, and $E = \{(x,y,z) | (x,y) \in D,\ 0 \leq z \leq 1 - x^2 - y^2\}$. S_1 is oriented downwards.

By Divergence Theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F} \, dV - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$. .... 4 points

If the orientation of S_1 is incorrect, -1 point only, i.e. still can get 4 points

Note that $\iiint_E \operatorname{div}\mathbf{F} \, dV = \iiint_E 6y^2 \, dV = I_1$. .... 3 points

Using polar coordinates $I_1 = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6(r\sin\theta)^2 r \, dz \, dr \, d\theta = \frac{\pi}{2}$. .... 2 points

On the other hand $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \langle 0, 0, -1 \rangle \, dS = \iint_D -e^{x^2+y^2} \, dA = I_2$, ... 3 points

$I_2 = \int_0^{2\pi} \int_0^1 -e^{r^2} r \, dr \, d\theta = -\pi(e-1)$. .... 3 points

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \pi(e - \frac{1}{2}).$$