## 1022微甲01-04班期中考解答和評分標準

1. (15%) (a) Let $\{b_n\}$ be a sequence of nonzero numbers such that $\lim_{n\to\infty} b_n = \infty$. Determine whether the series $\sum_{k=1}^{\infty}(b_{k+1} - b_k)$ and $\sum_{k=1}^{\infty}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right)$ are convergent or divergent. Explain your answer.

   (b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\left(n\sin\frac{1}{n} - 1\right)$ is absolutely convergent, conditionally convergent or divergent.

   (c) Find all values of $p$ such that the series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^p}$ converges conditionally.

**Solution:**

(a) Observe that

$$a - b = a - c + c - b$$

then apply it to the finite sums(1+1pts) and take limits(3pts).

(b) Observe that

$$n(\sin(\frac{1}{n}) - 1) \approxeq \frac{-1}{6n^2}$$

by taylor expansion of $\sin x$.(2pts)

Then using limit comparison.(3pts)

(c) For converge on p¿o, by alternating series test.(2pts)

For absolute converge on p¿1, by a consideration

$$p = 1 + 2\varepsilon$$

for any $\epsilon > 0$.

then

$$\frac{\ln(n)}{n^p} = \frac{\ln(n)}{n^{\varepsilon}}\frac{1}{n^{1+\varepsilon}}$$

And observe that

$$\frac{\ln(n)}{n^{\varepsilon}} < 1$$

for n large enough.

Now apply limit comparison with

$$\frac{1}{n^{1+\varepsilon}}$$

(3pts)

2. (10%) (a) Expand the function $f(x) = (8+x)^{\frac{1}{3}}$ as a power series centered at $x = 0$. (You must write out the general terms.) Find the radius of convergence.

(b) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n!}$.

**Solution:**

(a)

$$f(x) = 2(1+\frac{x}{8})^{\frac{1}{3}} = 2\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} (\frac{x}{8})^n$$

$$= 2 + \frac{x}{12} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!} \cdot (\frac{x}{8})^n$$

The radius of convergence: 8

配分:

- $2\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} (\frac{x}{8})^n$: 2分
- $2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!} \cdot (\frac{x}{8})^n$: 2分
- $2 + \frac{x}{12}$ and The radius of convergence is 8: 1分

(b)

$$\sum_{n=2}^{\infty} \frac{n^2+1}{n!} = \sum_{n=2}^{\infty} \frac{n(n-1)+n+1}{n!} = \sum_{n=2}^{\infty} \frac{1}{n-2!} + \frac{1}{n-1!} + \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{1}{n!} = (e) + (e-1) + (e-2)$$

$$= 3e - 3$$

配分:

- (e): 1分
- (e-1): 2分
- (e-2): 2分

3. (15%) Let $\{f_n\}$ be the Fibonacci sequence defined by $f_1 = f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Define $a_n = \dfrac{f_{n+1}}{f_n}$, $n \geq 1$.

(a) Show that $\{a_{2n}\}$ is decreasing while $\{a_{2n+1}\}$ is increasing and both $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist. Find the limits. (Hint. $\{a_n\}$ satisfies the recursive relation $a_{n+1} = 1 + \dfrac{1}{a_n}$, $n \geq 1$. Express $a_{n+2}$ in terms of $a_n$.)

(b) Find the radius of convergence of the power series $f(x) = \sum_{n=1}^{\infty} f_n x^n$.

**Solution:**

(a) We prove it by induction, given $f_1 = f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$
when $n = 1$,
$\Rightarrow a_1 = 1, a_2 = 2, a_3 = \dfrac{3}{2}, a_4 = \dfrac{5}{3}$.
$a_1 < a_3,\ a_2 > a_4$.
when $n = k$, suppose $a_{2k} > a_{2k+2}$,

$$1 + \frac{1}{a_{2k}} < 1 + \frac{1}{a_{2k+2}} \Rightarrow a_{2k+1} < a_{2k+3} \Rightarrow 1 + \frac{1}{a_{2k+1}} > 1 + \frac{1}{a_{2k+3}} \Rightarrow a_{2k+2} > a_{2k+4}$$

accroding to upper proof,

$n = k + 1$ is also hold for $[a_{2n}]$, so $[a_{2n}]$ is decreasing.

similar proof to $[a_{2n+1}]$, so $[a_{2n+1}]$ is increasing. (5 points)

On the other hand, clearly $f_n$ is increasing and all terms are positive,

$$a_n = \frac{f_{n+1}}{f_n} \geq 1 \Rightarrow 0 \leq a_{n+1} = 1 + \frac{1}{a_n} \leq 2, \forall n.$$

so $[a_{2n+1}]$ and $[a_{2n}]$ is bounded. (3 points)

Apply monotonic theorem, both limit exist, and

$$\lim_{n\to\infty} a_{2n} = L;\ \lim_{n\to\infty} a_{2n+1} = M$$

$$a_{2n+2} = 1 + \frac{1}{1 + \frac{1}{a_{2n}}} \Rightarrow L = 1 + \frac{1}{1 + \frac{1}{L}} \Rightarrow L = \frac{1+\sqrt{5}}{2}$$

similarly to $K = \dfrac{1+\sqrt{5}}{2}$.
finally we conclude that

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2} \quad \text{(3 points)}$$

(b) use ratio test,

$$\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| |x| = \lim_{n\to\infty} |a_n|\,|x| = \frac{1+\sqrt{5}}{2}\,|x|$$

since series is convergence,

$$\frac{1+\sqrt{5}}{2}\,|x| < 1 \Rightarrow |x| < \frac{\sqrt{5}-1}{2}$$

hence **R**$= \dfrac{\sqrt{5}-1}{2}$ (4 points)

4. (20%) A curve $C$ is defined by $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2 \rangle$, $t \geq 0$.

(a) Find the arc length function $s(t)$ with the starting point $(1, 0, 0)$.

(b) Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$ and the unit binormal vector $\mathbf{B}$.

(c) Find the curvature of $C$.

**Solution:**

(a) $r'(t) = (t\cos t, t\sin t, 2t)(1pt)$

$|r'(t)| = \sqrt{5}t(1pt)$

Hence $s(t) = \int_0^t |r'(s)|ds(1pt) = \int_0^t \sqrt{5}sds = \frac{\sqrt{5}t^2}{2}(2pt)$

(b) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}(1pt) = \frac{1}{\sqrt{5}}(\cos t, \sin t, 2)(2pt)$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}(1pt) = (-\sin t, \cos t, 0)(2pt)$

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)(2pt) = \frac{-1}{\sqrt{5}}(2\cos t, 2\sin t, -1)(2pt)$

(c) $\kappa(t) = |\frac{d\mathbf{T}}{ds}|(1pt) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}(1pt) = \frac{1}{5t}(3pt)$

5. (10%) Find the limit, if it exists, or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-x-y+1}{x^2+y^2-2x-2y+2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.

**Solution:**

(a)

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{xy-x-y+1}{x^2+y^2-2x-2y+2} &= \lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{(x-1)^2+(y-1)^2} \\ &= \lim_{(u,v)\to(0,0)} \frac{uv}{u^2+v^2}\end{aligned}$$

But

$$\begin{aligned} u=v, &\quad \lim_{(u,u)\to(0,0)} \frac{u^2}{u^2+u^2} = \frac{1}{2} \\ v=0, &\quad \lim_{(u,0)\to(0,0)} \frac{0}{u^2} = 0\end{aligned}$$

So the limit doesn't exist.
配分: 全對或全錯,計算錯誤扣1 2分.

(b)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left|\frac{x^3+y^3}{x^2+y^2}\right| &\leq \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\cdot|x| + \frac{y^2}{x^2+y^2}\cdot|y| \\ &\leq \lim_{(x,y)\to(0,0)} |x|+|y| \\ &= 0\end{aligned}$$

So

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

配分: 全對或全錯,少絕對直扣1分.

6. (20%) Let $f(x, y) = \int_1^{2y-x^2} e^{t^2}\,dt$.

(a) Find the rate of change of $f$ at the point $P(1,1)$ in the direction from $P$ to $Q(6,13)$.

(b) In what direction does $f$ have the maximum rate of change? What is this rate of change.

(c) Find the tangent plane and the normal line to the surface $S:\ z = f(x, y)$ at the point $(1, 1, 0)$.

(d) The sphere $x^2 + y^2 + z^2 = 2$ intersects $S$ in a curve $C$. Find the equations for the tangent line to $C$ at the point $(1, 1, 0)$.

**Solution:**

We calculate $\nabla f$ first. (Maximum 5 points for finding $\nabla f$ correctly.)

$\nabla f = (-2x \cdot e^{(2y-x^2)^2}, 2e^{(2y-x^2)^2})$

$= 2e^{(2y-x^2)^2}(-x, 1)$.

(a) $\overrightarrow{PQ} = (6, 13) - (1, 1) = (5, 12)$

$\vec{u} = \frac{1}{\sqrt{5^2 + 12^2}}(5, 12) = (\frac{5}{13}, \frac{12}{13})$ (1 point)

$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u} = (-2e, 2e) \cdot (\frac{5}{13}, \frac{12}{13}) = \frac{14}{13}e.$ (2 points)

(b) $\vec{u} = (\frac{-x}{\sqrt{x^2+1}}, \frac{1}{\sqrt{x^2+1}})$ (2 points)

$|\nabla f(x, y)| = 2e^{(2y-x^2)^2}\sqrt{x^2 + 1}.$ (2 points)

(c)Normal line: $x = 1 - 2et,\quad y = 1 + 2et,\quad z = t,\quad t \in R$ (1 point)

Tangent plame:

$z - 0 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$ (1 point)

$= -2e(x - 1) + 2e(y - 1)$

$= -2ex + 2ey.$ (1 point)

(d) Let $g(x, y, z) = x^2 + y^2 + z^2 - 2$

$\nabla g(1, 1, 0) = (2, 2, 0)$ (1 point)

$\nabla S(1, 1, 0) = (-2e, 2e, -1)$ (1 point)

$\nabla S \times \nabla g = (1, -1, -4e)$ (2 points)

$\Rightarrow x = 1 + t,\quad y = 1 - t,\quad z = -4et,\quad t \in R$ (1 point)

7. (10%) Let $f(x,y)=\sin x\cos(x+y)$ and $D=\left\{(x,y)|0\le x\le \frac{\pi}{2}, 0\le y\le \frac{\pi}{2}\right\}$. Classify all the critical points of $f$ on $D$.

**Solution:**

$f_x(x,y)=\cos x\cos(x+y)-\sin x\sin(x+y)=\cos(2x+y)$ and $f_y(x,y)=-\sin x\sin(x+y)=\dfrac{\cos(2x+y)-\cos y}{2}$. Then let $f_x(x,y)=f_y(x,y)=0\Rightarrow \cos(2x+y)=\cos y=0\Rightarrow y=\dfrac{\pi}{2}$ and $x=0$ or $\dfrac{\pi}{2}$. So critical points of f on D are $(0,\dfrac{\pi}{2})$ and $(\dfrac{\pi}{2},\dfrac{\pi}{2})$. Then $f_{xx}(x,y)=-2\sin(2x+y)$, $f_{xy}(x,y)=-\sin(2x+y)$ and $f_{yy}(x,y)=-\sin x\cos(x+y)$. $D(0,\dfrac{\pi}{2})=-2\cdot 0-(-1)^2=-1<0$ and $D(\dfrac{\pi}{2},\dfrac{\pi}{2})=2\cdot 1-(1)^2=1>0$. Hence $(0,\dfrac{\pi}{2})$ is saddle point and $(\dfrac{\pi}{2},\dfrac{\pi}{2})$ is local minimum.

評分標準：
$f$ 對 $x$ 和 $y$ 的偏微分有算出來的有 2 分

在 $D$ 上解方程式有解出來的有 4 分(包含求出 critical point)

算出二階偏微分的有 2 分

帶入判別式分出 critical point 是哪一種的有 2 分

8. (10%) Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 \leq 1$.

**Solution:**

Let $f = xy + z^2$ and $g = x^2 + y^2 + (z - \frac{1}{2})^2 - 1$.

For the points inside the ball $x^2 + y^2 + (z - \frac{1}{2})^2 \leq 1$, consider critical points of $f$:

$$\nabla f = \langle y, x, 2z \rangle = 0 \Rightarrow (x, y, z) = (0, 0, 0).$$

We have $f(0, 0, 0) = 0$. (1 pt)

For the points on the boundary, that is, points satisfy $g = 0$, consider $\nabla f = \lambda \nabla g$:

$$\begin{cases} y = \lambda(2x) & (1) \\ x = \lambda(2y) & (2) \\ 2z = \lambda(2(z - \frac{1}{2})) & (3) \\ x^2 + y^2 + (z - \frac{1}{2})^2 = 1 & (4) \end{cases} . \qquad \text{(2 pts)}$$

From (1)(2), $x = \lambda(2\lambda(2x)) \Rightarrow x(4\lambda^2 - 1) = 0 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.

(i) $x = 0 \Rightarrow y = 0$, from (4), $0 + 0 + (z - \frac{1}{2})^2 = 1 \Rightarrow z = -\frac{1}{2}$ or $\frac{3}{2}$.

$f(0, 0, -\frac{1}{2}) = \frac{1}{4}$, $f(0, 0, \frac{3}{2}) = \frac{9}{4}$. (2pts)

(ii) $\lambda = \frac{1}{2}$, from (1)(2), $x = y$; from (3), $z = -\frac{1}{2}$.

$\Rightarrow$ From (4), $x^2 + x^2 + (-\frac{1}{2} - \frac{1}{2})^2 = 1 \Rightarrow x = y = 0$.

$f(0, 0, -\frac{1}{2}) = \frac{1}{4}$. (2pts)

(iii) $\lambda = -\frac{1}{2}$, from (1)(2), $x = -y$; from (3), $z = \frac{1}{6}$.

$\Rightarrow$ From (4), $x^2 + x^2 + (\frac{1}{6} - \frac{1}{2})^2 = 1 \Rightarrow x = \pm\frac{2}{3}$ and $y = \mp\frac{2}{3}$.

$f(\frac{2}{3}, -\frac{2}{3}, \frac{1}{6}) = f(-\frac{2}{3}, \frac{2}{3}, \frac{1}{6}) = -\frac{15}{36}$. (2pts)

Therefore, the maximum is $\frac{9}{4}$ and the minimum is $-\frac{15}{36}$. (1pt)