1. (10%)

   (a) Evaluate the limit

$$I_m = \lim_{n\to\infty}\sum_{i=1}^{nm}\frac{i^2n^3}{n^6+i^6} = \lim_{n\to\infty}\left(\frac{n^3}{n^6+1^6}+\frac{2^2n^3}{n^6+2^6}+\cdots+\frac{(nm-1)^2n^3}{n^6+(nm-1)^6}+\frac{(nm)^2n^3}{n^6+(nm)^6}\right),$$

   where m is a positive integer.

   (b) Compute $\lim_{m\to\infty} I_m$.

Solution:

(a)(6%)

The original equation $= \frac{1}{n}\Sigma_{i=1}^{nm}\frac{i^2n^4}{n^6+i^6}$

$= \frac{1}{n}\Sigma_{i=1}^{nm}\frac{(\frac{i}{n})^2}{1+(\frac{i}{n})^6}$

By the definition of Riemann sum,

$= \int_0^m \frac{x^2}{1+x^6}\mathrm{d}x$ (3 points)

$= \frac{1}{3}\int_0^m \frac{dx^3}{1+(x^3)^2}$ (1 point)

$= \frac{1}{3}\tan^{-1}(x^3)\,|_0^m + C$

$= \frac{1}{3}\tan^{-1}(m^3) + C$ (2 points)

Therefore, $I_m = \frac{1}{3}\tan^{-1}(m^3)$, where m is a positive integer .

ps. Other methods for solving $\int_0^m \frac{x^2}{1+x^6}\mathrm{d}x$ are permitted.

(b)(4%)

$$\lim_{m\to\infty} I_m = \lim_{m\to\infty}\frac{1}{3}\tan^{-1}(m^3) = \frac{1}{3}\cdot\frac{\pi}{2} = \frac{\pi}{6}.$$

2. (15%) Evaluate the integral.

(a) $\int_0^1 x^2 \sqrt[3]{1-x}dx.$

(b) $\int x\sqrt{3-2x-x^2}dx.$

Solution:

(a)(8%)

Let $u = \sqrt[3]{1-x}$, and we can find that $u^3 = 1 - x$ and $dx = -3u^2\mathrm{d}u$.

$$\text{The original equation} = \int_1^0 (1-u^3)^2 \cdot u \cdot (-3u^2)\mathrm{d}u$$

$$= -3\int_1^0 (u^9 - 2u^6 + u^3)\mathrm{d}u \qquad \text{(4 points)}$$

$$= -3(\frac{1}{10}u^{10} - \frac{2}{7}u^7 + \frac{1}{4}u^4) \Big|_1^0 \qquad \text{(2 points)}$$

$$= \frac{27}{140}. \qquad \text{(2 points)}$$

ps. Other methods for solving this integral are permitted.

(b)(7%)

The original equation $= \int x\sqrt{4-(x+1)^2}\ \mathrm{d}x$ ....Equation(1)

Let $x + 1 = 2\sin\theta$, and we can find that $\mathrm{d}x = 2\cos\theta\ \mathrm{d}\theta$ ....Equation(2)

Substitute Equation(2) into Equation(1), (3 points for Trigonometric substitution)

and we have $\int (2\sin\theta - 1)\cdot 2\cos\theta \cdot 2\cos\theta\ \mathrm{d}\theta.$ (3 points)

$$= \frac{-8}{3}\cos^3\theta - 2\theta - \sin 2\theta + C$$

$$= \frac{-1}{3}(3-2x-x^2)^{\frac{3}{2}} - 2\sin^{-1}(\frac{x+1}{2}) - \frac{x+1}{2}\sqrt{3-2x-x^2} + C. \qquad \text{(1 point)}$$

ps. Other methods for solving this integral are permitted.

3. (20%) Evaluate the integral.

(a) $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx.$

(b) $\displaystyle\int_1^\infty \frac{x^2-3}{(x^2-2x+3)(x^2+2x+3)}dx.$

Solution:

(a) (8%)

Sol. (1)

$$\begin{aligned}\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx &= \int \frac{\frac{1}{2}\sin 2x}{(\frac{1-\cos x}{2})^2+(\frac{1+\cos x}{2})^2}dx\\ &\text{with } u=\cos 2x,\ du=-2\sin 2x dx\\ &= -\frac{1}{2}\int \frac{1}{1+u^2}du\\ &= -\frac{1}{2}\tan^{-1} u + C\\ &= -\frac{1}{2}\tan^{-1}(\cos 2x)+C\end{aligned}$$

Sol. (2)

$$\begin{aligned}\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx &= \int \frac{\tan x\sec^2 x}{1+\tan^4 x}dx\\ &\text{with } u=\tan^2 x,\ du=2\tan x\sec^2 x dx\\ &= \int \frac{\frac{1}{2}du}{1+u^2}\\ &= \frac{1}{2}\tan^{-1} u + C\\ &= \frac{1}{2}\tan^{-1}(\tan^2 x)+C\end{aligned}$$

Sol. (3)

$$\begin{aligned}\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx &= \int \frac{\sin x\cos x}{\sin^4 x+(1-\sin^2 x)^2}dx\\ &\text{with } u=\sin^2 x,\ du=2\sin x\cos x dx\\ &= \int \frac{\frac{1}{2}du}{u^2+(1-u)^2}\\ &= \frac{1}{2}\int \frac{1}{2u^2-2u+1}du\\ &= \frac{1}{2}\int \frac{2}{(2u-1)^2+1}du\\ &= \frac{1}{2}\tan^{-1}(2u-1)+C\\ &= \frac{1}{2}\tan^{-1}(2\sin^2 x-1)+C\end{aligned}$$

Grading policy:
(1) 4% for any proper change of variables (including correct du).
(2) 4% for correct integration. -1% if C is missing.

(b) (12%)
(1) Partial fraction (5%)

$$\begin{aligned}\text{Define } f(x) &= \frac{x^2-3}{(x^2-2x+3)(x^2+2x+3)} \\ &= \frac{Ax+B}{x^2-2x+3} + \frac{Cx+D}{x^2+2x+3} \qquad \text{(2\% for the form)}\end{aligned}$$

$$\Rightarrow \begin{cases} x^3: & A+C & =0 \\ x^2: & 2A+B-2C+D & =1 \\ x^1: & 3A+2B+3C-2D & =0 \\ x^0: & 3B+3D & =-3 \end{cases}$$

$$\Rightarrow \begin{cases} A & =\frac{1}{2} \\ B & =-\frac{1}{2} \\ C & =-\frac{1}{2} \\ D & =-\frac{1}{2} \end{cases}$$

$$\therefore f(x) = \frac{1}{2}\left(\frac{x-1}{x^2-2x+3} - \frac{x+1}{x^2+2x+3}\right) \qquad \text{(3\%)}$$

(2) Integration (4 %)

$$\begin{aligned}&\int \frac{1}{2}\left(\frac{x-1}{x^2-2x+3} - \frac{x+1}{x^2+2x+3}\right)dx \\ &= \int \frac{1}{4}\left(\frac{2x-2}{x^2-2x+3} - \frac{2x+2}{x^2+2x+3}\right)dx \\ &= \frac{1}{4}\left(\ln\left|x^2-2x+3\right| - \ln\left|x^2+2x+3\right|\right) + C \\ &= \frac{1}{4}\ln\left|\frac{x^2-2x+3}{x^2+2x+3}\right| + C\end{aligned}$$

(3) Improper integral (3 %)

$$\begin{aligned}&\int_1^\infty f(x)dx \\ &= \lim_{b\to\infty}\int_1^b \frac{1}{4}\left(\frac{2x-2}{x^2-2x+3} - \frac{2x+2}{x^2+2x+3}\right)dx \\ &= \lim_{b\to\infty}\left[\frac{1}{4}\ln\left|\frac{x^2-2x+3}{x^2+2x+3}\right|\right]_1^b \\ &= \lim_{b\to\infty}\frac{1}{4}\ln\left|\frac{b^2-2b+3}{b^2+2b+3}\right| - \frac{1}{4}\ln\left|\frac{1-2+3}{1+2+3}\right| \\ &= \frac{1}{4}\ln 1 - \frac{1}{4}\ln\frac{1}{3} \\ &= \frac{1}{4}\ln 3\end{aligned}$$

(Upper limit: 2%; lower limit: 1%.)

Grading policy:
The three parts are credited independently, e.g., you will still get full 4% in part (2) if the integration process is correct based on your result in (1), even though your result in (1) may be incorrect.

4. (10%) A function $y = y(x)$ satisfies the equation

$$y(x) = x + \int_0^{x^2} \left(x - y(\sqrt{t}) - \sqrt{t} - \frac{1}{2\sqrt{t}} + 1\right) e^t dt,\ x \geq 0.$$

(a) Find a differential equation with initial condition for y.

(b) Solve the differential equation.

Solution:

$$y = x + x\int_0^{x^2} e^t dt - (\int_0^{x^2} (y(\sqrt{t}) + \sqrt{t} + \frac{1}{2\sqrt{t}} - 1)e^t dt)\ (1\ point)$$

Differentiate both side,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \int_0^{x^2} e^t dt + x \cdot 2xe^{x^2} - (y(x) + x + \frac{1}{2x} - 1) \cdot e^{x^2} \cdot 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xe^{x^2}(1 - y(x))\ or\ \frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y(x) = 2xe^{x^2}\ (3\ points)$$

and boundary condition $y(0) = 0$ (1 point) you may choose intergal factor or separable method,

- intergal fatcor:

$$\int 2xe^{x^2} dx = e^{e^{x^2}}\ (2\ points)$$

$$\Rightarrow (e^{e^{x^2}} y)' = 2xe^{x^2} e^{e^{x^2}}$$

$$\Rightarrow e^{e^{x^2}} y(x) = \int 2xe^{x^2} e^{e^{x^2}} dx = e^{e^{x^2}} + C_1\ (1\ point)$$

- separable mentod:

$$\int \frac{1}{1-y} dy = \int 2xe^{x^2} dx$$

$$\Rightarrow -\ln(1-y) = e^{x^2} + C_2$$

$$\Rightarrow y(x) = 1 - e^{-e^{x^2} - C_2}$$

apply boundary condition $y(0) = 0$,

$$\Rightarrow y(x) = 1 - e^{1-e^{x^2}}\ (2\ points)$$

if you make a mistake at the first step:

$$y = 1 + (x - y(x) - x - \frac{1}{2x} + 1) \cdot 2xe^{x^2}$$

we'll give you 2 points , write correct boundary condition, giving 1 point and finally you calculate integral factor, giving 2 points; totally 5 points.

5. (10%)

(a) If the infinite curve $y = e^{-x}$, $x \geq 0$, is rotated about the x-axis, find the area of the resulting surface.

(b) Find the arc length of the infinite curve with polar equation $r = \theta^{-1}$, $\theta \geq 1$.

Solution:

(a)
The area of surface is

$$S = 2\pi \int_0^\infty e^{-x}\sqrt{1+(-e^{-x})^2}dx(2pt)$$

$$= 2\pi \int_1^0 -\sqrt{1+u^2}du$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 y}\sec^2 ydy$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sec^3 ydy(1pt)$$

$$= \pi[\log(\sqrt{2}+1)+\sqrt{2}](2pt)$$

(b)
The arc length is

$$Ł = \int_1^\infty \sqrt{r^2+(\frac{dr}{d\theta})^2}d\theta$$

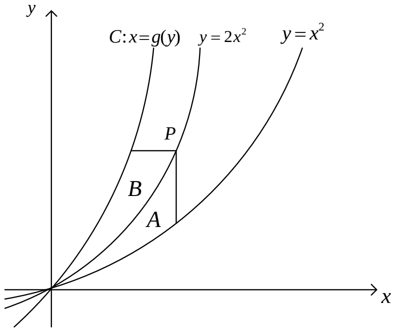
$$\int_1^\infty \sqrt{\theta^{-2}+\theta^{-4}}d\theta(2pt)$$

$$\int_1^\infty \frac{1}{\theta^{-2}}\sqrt{1+\theta^2}d\theta$$

$$> \int_1^\infty \frac{1}{\theta^{-2}}\sqrt{\theta^2}d\theta$$

$$= \int_1^\infty \frac{1}{\theta}d\theta = \infty(3pt)$$

6. (20%) The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the area of B is twice the area of A.



(a) Find an equation $x = g(y)$ for C.

(b) Let R be the region bounded by the curve C, $y = x^2$, $x = 2$ and $y = 8$. Find the volume of the solid obtained by rotating R about the x-axis.

(c) Find the y-coordinate of the centroid of R.

Solution:

(a) Assume $P(t, 2t^2)$ is a point on the curve $y = 2x^2$.

The area of $A = \int_0^t (2x^2 - x^2)\mathrm{d}x = \int_0^t x^2\mathrm{d}x = \frac{1}{3}x^3|_0^t = \frac{1}{3}t^3$

$$\text{The area of } B = \int_0^{2t^2} (\sqrt{\frac{y}{2}} - g(y))\mathrm{d}y = \frac{1}{\sqrt{2}} \cdot \frac{2}{3}y^{\frac{3}{2}}|_0^{2t^2} - \int_0^{2t^2} g(y)\mathrm{d}y$$

$$= \frac{\sqrt{2}}{3}(2\sqrt{2}t^3) - \int_0^{2t^2} g(y)\mathrm{d}y = \frac{4}{3}t^3 - \int_0^{2t^2} g(y)\mathrm{d}y$$

$B = 2A \Rightarrow \int_0^{2t^2} g(y)\mathrm{d}y = \frac{2}{3}t^3$ 3 pts

$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2t^2} g(y)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{2}{3}t^3)$ 2 pts

By the Fundamental Theorem of Calculus, $g(2t^2) \cdot 4t = 2t^2 \Rightarrow g(2t^2) = 2t$

Let $y = 2t^2 \Rightarrow t = \sqrt{\frac{y}{2}} \Rightarrow g(y) = \frac{1}{2}\sqrt{\frac{y}{2}}$. 3pts

That is, $y = 8x^2$.

(b) Note that the point $(2, 8)$ is on the curve $y = 2x^2$.

Sol 1.

$$V = \int_0^8 2\pi y(\sqrt{\frac{y}{2}} - \frac{1}{2}\sqrt{\frac{y}{2}})\mathrm{d}y + \int_0^2 \pi((2x^2)^2 - (x^2)^2)\mathrm{d}x \quad \text{2pts}$$

$$= 2\pi \int_0^8 \frac{1}{2\sqrt{2}}y^{\frac{3}{2}}\mathrm{d}y + \pi\int_0^2 3x^4\mathrm{d}x$$

$$= \frac{1}{\sqrt{2}}\pi(\frac{2}{5}y^{\frac{5}{2}})|_0^8 + \pi(3 \cdot \frac{1}{5}x^5)|_0^2$$

$$= \frac{1}{\sqrt{2}}\pi \cdot \frac{2}{5}(128\sqrt{2}) + \pi(\frac{3}{5} \cdot 32)$$

$$= \frac{256}{5}\pi + \frac{96}{5}\pi = \frac{352}{5}\pi \quad \text{4pts}$$

Sol 2. On the curve C, when $y = 8 \Rightarrow x = \frac{1}{2}\sqrt{\frac{8}{2}} = 1$

$$\begin{aligned} V &= \int_0^1 \pi((8x^2)^2-(x^2)^2)\mathrm{d}x + \int_1^2 \pi((8)^2-(x^2)^2)\mathrm{d}x \qquad \text{2pts} \\ &= \pi\int_0^1 63x^4\mathrm{d}x + \pi\int_1^2 (64-x^4)\mathrm{d}x \\ &= \pi(\frac{63}{5}x^5)|_0^1 + \pi(64x-\frac{1}{5}x^5)|_1^2 \\ &= \frac{63}{5}\pi + (128-\frac{32}{5}-64+\frac{1}{5})\pi = \frac{352}{5}\pi \qquad \text{4pts} \end{aligned}$$

(c)

$$\begin{aligned} \text{The area of } R &= \int_0^1 (8x^2-x^2)\mathrm{d}x + \int_1^2 (8-x^2)\mathrm{d}x \\ &= \frac{7}{3}x^3|_0^1 + (8x-\frac{1}{3}x^3)|_1^2 = \frac{7}{3}+16-\frac{8}{3}-8+\frac{1}{3} = 8 \qquad \text{3pts} \end{aligned}$$

or The area of $R = 3\int_0^2 (2x^2-x^2)\mathrm{d}x = 3(\frac{1}{x}x^3)|_0^2 = 3\cdot\frac{8}{3} = 8$ 3pts

Sol 1. By the Pappus Theorm, $V = 2\pi\bar{y}A_{rea}$.
Hence $\bar{y} = \frac{V}{2\pi A_{rea}} = (\frac{352}{5}\pi)/(2\pi\cdot 8) = \frac{22}{5}$. 3pts

Sol 2.

$$\begin{aligned} \text{moment} &= \int_0^1 \frac{1}{2}((8x^2)^2-(x^2)^2)\mathrm{d}x + \int_1^2 \frac{1}{2}((8)^2-(x^2)^2)\mathrm{d}x \\ &= \frac{1}{2}\cdot\frac{63}{5}x^5|_0^1 + \frac{1}{2}(64x-\frac{1}{5}x^5)|_1^2 \\ &= \frac{1}{2}\cdot\frac{63}{5} + \frac{1}{2}(128-\frac{32}{5}-64+\frac{1}{5}) = \frac{176}{5} \end{aligned}$$

Therefore $\bar{y} = \frac{\text{moment}}{A_{rea}} = \frac{176}{5}\cdot\frac{1}{8} = \frac{22}{5}$. 3pts

7. (15%) Let the curve C defined by $\begin{cases} x = t^2 \\ y = \dfrac{t^3}{3} - t \end{cases}$, $t \in \mathbb{R}$.

(a) Find the point P where the curve intersects itself.

(b) Find the equation of the tangent lines at the point P.

(c) For which values of t is the curve increasing?

(d) For which values of t is the curve concave upward?

(e) Sketch the curve C.

Solution:

(a)
(3分)
假設C上的兩點為$(t_1^2, \frac{t_1^3}{3} - t_1)$，$(t_2^2, \frac{t_2^3}{3} - t_2)$。
由P為自交點我們可以得到兩條方程式:

$$t_1^2 = t_2^2 \text{ (方程式一)}$$

和

$$\frac{t_1^3}{3} - t_1 = \frac{t_2^3}{3} - t_2 \text{ (方程式二)}$$

由方程式一可以得到$(t_1 - t_2)(t_1 + t_2) = 0$，即$t_1 = t_2$或$t_1 = -t_2$
($t_1 = t_2$不合，因為自交點的定義是不同的參數t會得到同一點)。
因此$t_1 = -t_2$。接著將$t_1 = -t_2$代入方程式二，得到

$$\frac{-t_2^3}{3} + t_2 = \frac{t_2^3}{3} - t_2$$

解出來可得到$t_2 = 0$或$t_2 = \pm\sqrt{3}$($t_2 = 0$不合，因為$t_2 = 0$會導致$t_1 = t_2$)。
令$t_2 = \sqrt{3}$(所以$t_1 = -\sqrt{3}$)，就可以得到$x = 3$，$y = 0$，所以$P = (3, 0)$ (註:令$t_2 = -\sqrt{3}$也可以算出相同答案，它們是兩種不同C的參數化)。

(b)
(3分)
曲線C上的每點切線斜率:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dx}} = \frac{t^2 - 1}{2t}$$

由(a)知道切點為$(3, 0)$，我們可以假設切線方程式為$y - 0 = \dfrac{t^2 - 1}{2t}(x - 3)$，接著把從(a)算出來的$t_1$，$t_2$代入該方程式:

$$t = t_2 = \sqrt{3} \Rightarrow y - 0 = \frac{(\sqrt{3})^2 - 1}{2\sqrt{3}}(x - 3) \Rightarrow y = \frac{1}{\sqrt{3}}(x - 3)$$

$$t = t_1 = -\sqrt{3} \Rightarrow y - 0 = \frac{(-\sqrt{3})^2 - 1}{2(-\sqrt{3})}(x - 3) \Rightarrow y = \frac{-1}{\sqrt{3}}(x - 3)$$

因此 $y = \dfrac{1}{\sqrt{3}}(x - 3)$ 和 $y = \dfrac{-1}{\sqrt{3}}(x - 3)$ 即為該點切線方程式。

(c)
(3分)
由一階檢定可知 $\dfrac{dy}{dx} \geq 0 \Rightarrow C$ 遞增，所以

$$\frac{dy}{dx} \geq 0 \iff \frac{t^2 - 1}{2t} \geq 0 \iff 2t(t^2 - 1) \geq 0 \text{但} t \neq 0 \iff t(t - 1)(t + 1) \geq 0$$

$$但t \neq 0 \Longleftrightarrow t \geq 0或 -1 \leq t < 0。$$

因此當 $t \geq 0$或 $-1 \leq t < 0$ 時，曲線 C 遞增。

(d)

(3分)

由二階檢定可知 $\frac{d^2y}{dx^2} > 0 \Rightarrow C$ 開口向上，所以

$$\frac{d^2y}{dx^2} > 0 \Longleftrightarrow \frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} > 0 \Longleftrightarrow \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{(2t)^2-2(t^2-1)}{(2t)^2}}{2t} > 0$$

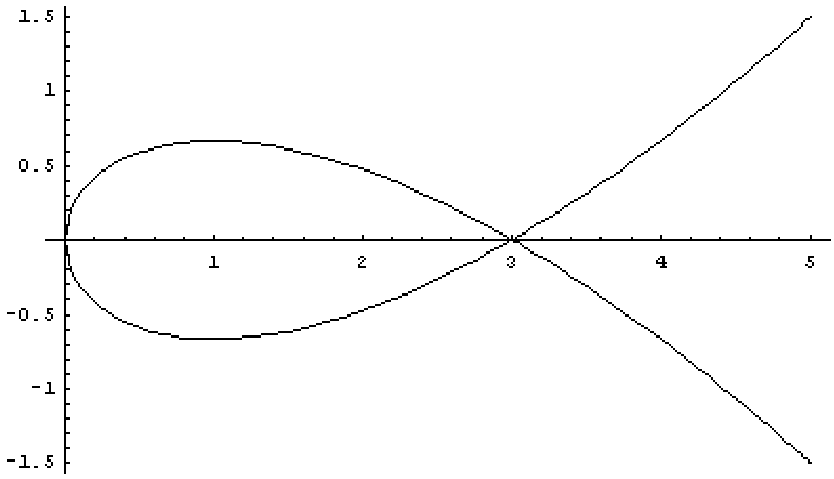
$$\Longleftrightarrow \frac{t^2+1}{2t^3} > 0 \Longleftrightarrow t^3 > 0 \Longleftrightarrow t > 0$$

因此當 $t > 0$ 時，曲線 C 開口向上。

(e)

(3分)

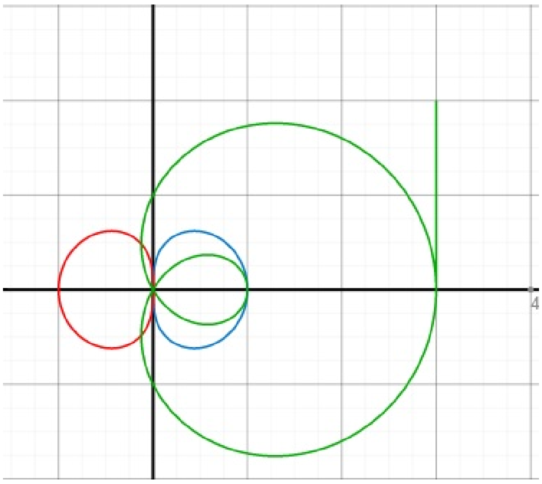
利用(a)(b)(c)(d)來畫圖(註:可以試著把參數式寫成x和y的表示式:$y^2 = \frac{x^3}{9} - \frac{2x^2}{3} + x$，不難發現它是對$x$軸對稱)



8. (10%)

(a) Find the points of intersection of the curves $r = 1 + 2\cos\theta$ and $r^2 = \cos\theta$.

(b) Find the area of the region in the second quadrant that lies inside $r^2 = \cos\theta$ and outside $r = 1 + 2\cos\theta$.



Solution:

(a)
Claim the intersection points for for polar coordinate are

$$[0, 0], [1, 0], [\frac{1}{2}, \pm\cos^{-1} -\frac{1}{4}].$$

Equally, for Cartesian coordinate, they are

$$(0, 0), (1, 0), (-\frac{1}{8}, \pm\frac{\sqrt{15}}{8}).$$

Part 1: the origin.
Claim $[0, 0]$ is an intersection point.
For $r = 1 + 2\cos\theta$, there is $\theta = \frac{2\pi}{3}$ such that $r = 1 + 2\cos\theta = 1 + 2\cos\frac{2\pi}{3} = 0$.
For $r^2 = \cos\theta$, there is $\theta = \frac{\pi}{2}$ such that $r^2 = \cos\theta = \cos\frac{\pi}{2} = 0$.
We find that $[0, 0]$ is on the both curves so is an intersection point.

Part 2: other points.
A point $[r_0, \theta_0]$ on $r = 1 + 2\cos\theta$ is also on $r^2 = \cos\theta$ if and only if
Case 1: $[r_0, \theta_0]$ satisfies $r^2 = \cos\theta$.
Case 2: $[-r_0, \theta_0 + \pi]$ satisfies $r^2 = \cos\theta$.

Case 1:
We have

$$(1 + 2\cos\theta_0)^2 = r_0^2 = \cos\theta_0$$

$$1 + 3\cos\theta_0 + 4\cos^2\theta_0 = 0$$

But there is no solution for such θ_0.

Case 2:
We have

$$(1 + 2\cos\theta_0)^2 = r_0^2 = (-r_0)^2 = \cos(\theta_0 + \pi) = -\cos\theta_0$$

$$1 + 5\cos\theta_0 + 4\cos^2\theta_0 = 0$$

We find $\cos\theta_0 = -1, -\dfrac{1}{4}$.
When $\cos\theta_0 = -1$, $\theta_0 = \pi$ and $r_0 = 1 + 2\cos\theta_0 = -1$, so

$$[r_0, \theta_0] = [-1, \pi] = [1, 0].$$

When $\cos\theta_0 = -\dfrac{1}{4}, \theta_0 = \pm\cos^{-1} -\dfrac{1}{4}$ and $r_0 = 1 + 2\cos\theta_0 = \dfrac{1}{2}$, so

$$[r_0, \theta_0] = [\frac{1}{2}, \pm\cos^{-1} -\frac{1}{4}].$$

In this, the arc cosine function is denoted as $\cos^{-1}$.

(b)
Observe that the boundary of the desired region are x-axis,

$$r = -\sqrt{\cos\theta} \text{ for } \theta \text{ from } \pi + \cos^{-1} -\frac{1}{4} \text{ to } 2\pi,$$

and

$$r = 1 + 2\cos\theta \text{ for } \theta \text{ from } \cos^{-1} -\frac{1}{4} \text{ to } \frac{2\pi}{3}.$$

Therefore, the area of the region is

$$\int_{\pi+\cos^{-1}-\frac{1}{4}}^{2\pi} \frac{1}{2}(-\sqrt{\cos\theta})^2 d\theta - \int_{\cos^{-1}-\frac{1}{4}}^{\frac{2\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_{\pi+\cos^{-1}-\frac{1}{4}}^{2\pi} \cos\theta d\theta - \frac{1}{2}\int_{\cos^{-1}-\frac{1}{4}}^{\frac{2\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta d\theta$$

$$= \frac{1}{2}\sin\theta|_{\pi+\cos^{-1}-\frac{1}{4}}^{2\pi} - \frac{1}{2}\left(\theta + 4\sin\theta + \sin 2\theta + 2\theta\right)_{\cos^{-1}-\frac{1}{4}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{2}\frac{\sqrt{15}}{4} - \frac{1}{2}\left(2\sqrt{3} - \frac{\sqrt{3}}{2} + 2\pi - \sqrt{15} + \frac{\sqrt{15}}{8} - 3\cos^{-1} -\frac{1}{4}\right)\Bigg|$$

$$= \frac{9\sqrt{15}}{16} - \frac{3\sqrt{3}}{4} - \pi + \frac{3}{2}\cos^{-1} -\frac{1}{4}$$

Criteria:
(a)
There are four criteria:
1. Explain that the origin [0, 0] point is an intersection point. 1 point.
2. Use the technique of polar coordinate to find that [1, 0] is an intersection point. 1 point.
3. Solve and find 2 intersection points for case of arc cosine. 1 point for each.
There are other situations:
1. Only write down the correct answer. 1 point.
2. Write some meaning computation but no identify any intersection point. 1 point.

(b)
1. Write down the correct integral range. 1 point for each.
2. Write down the correct integrated functions.1 point .
3. Compute this integral for first part: $\cos\theta/2$. 1 point .
4. Compute this integral for second part: $(1 + 2\cos\theta)^2/2$. 1 point .
5. Find the final answer. 1 point.