1. (15 points) Let $s_k = \sum_{j=1}^{k} \frac{1}{j}$, $k = 1,\ 2,\ \cdots$, and $A(x) = \sum_{k=1}^{\infty} s_k x^k$.
   (a) Find the interval of convergence of $A(x)$.
   (b) Express $A(x)$ in terms of elementary functions by comparing $A(x)$ and $xA(x)$.

Solution:

(a)

Using ratio test, (knowing how to use ratio test or root test in the correct way earn 1 point)

$$\frac{s_{k+1}}{s_k} = 1 + \frac{(\frac{1}{k+1})}{s_k} \to 1 \text{ as } k \to \infty,$$

thus the radius of convergence of $A(x)$ is 1. (having computed the radius of convergence earn 3 points)
(The radius of convergence can also be calculated using root test.

$$\sqrt[k]{s_k} = \sqrt[k]{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}} \leq \sqrt[k]{1 + 1 + \cdots + 1} = \sqrt[k]{k}$$
$$\sqrt[k]{s_k} = \sqrt[k]{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}} \geq \sqrt[k]{\frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k}} = 1$$

Since

$$\sqrt[k]{k} = e^{\frac{\ln k}{k}}$$

and

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{(\frac{1}{t})}{1} = 0 \text{ by l'hospital rule.}$$

Thus

$$\lim_{k\to\infty} \sqrt[k]{k} = 1$$

and by squeezing

$$\lim_{k\to\infty} \sqrt[k]{s_k} = 1.$$

)

Note that

$$|s_k(\pm 1)^k| = s_k \to \infty \neq 0 \text{ as } k \to \infty,$$

hence $A(x)$ diverges at $x = \pm 1$. The interval of convergence of $A(x)$ is $(-1, 1)$. (obtaining the endpoints behavior earn 1 point)

(b)

For $x \in (-1, 1)$,

$$A(x) = \sum_{k=1}^{\infty} s_k x^k = x + \sum_{k=2}^{\infty} s_k x^k$$
$$xA(x) = \sum_{k=1}^{\infty} s_k x^{k+1} = \sum_{k=2}^{\infty} s_{k-1} x^k.$$

Subtracting them, we obtain

$$(1-x)A(x) = x + \sum_{k=2}^{\infty} (s_k - s_{k-1}) x^k = x + \sum_{k=2}^{\infty} \frac{1}{k} x^k = \sum_{k=1}^{\infty} \frac{1}{k} x^k. \text{ (5 points)}$$

But

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x),$$

So

$$A(x) = -\frac{\ln(1-x)}{1-x}. \text{ (5 points)}$$

2. (15 points)

(a) Expand $f(x) = (x-1)\ln(1+3x)$ in powers of $x-1$.

(b) For what values of x is the above expansion valid?

(c) Find the sum $\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{4}\right)^k$.

Solution:

(a) $f(x) = (x-1)\ln(1+3x) = (x-1)\ln[3(x-1)+4] = (x-1)\ln[4(1+\frac{3}{4}(x-1))] = (x-1)[\ln 4+\ln[1+\frac{3}{4}(x-1)]] = (x-1)[\ln 4 + \Sigma_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[\frac{3}{4}(x-1)]^k]$

(b) $-1 < \frac{3}{4}(x-1) \le 1$, which is say if $\frac{-1}{3} < x \le \frac{7}{3}$

(c) $f(0) = 0 = -\ln 4 + \Sigma_{k=1}^{\infty}\frac{1}{k}(\frac{3}{4})^k$.
$\Rightarrow \Sigma_{k=1}^{\infty}\frac{1}{k}(\frac{3}{4})^k = \ln 4$.

3. (15 points) Let $\mathcal{C}$ be the curve given by $\boldsymbol{r}(t) = \frac{2}{3}(1+t)^{\frac{3}{2}}\boldsymbol{i} + \frac{2}{3}(1-t)^{\frac{3}{2}}\boldsymbol{j} + at\boldsymbol{k}$, $t \in (-1,1)$, $a \in \mathbb{R}\setminus\{0\}$.

(a) Find the length $s(b)$ of the curve from $t=0$ to $t=b \in (0,1)$.

(b) Find the unit tangent, the principal normal, and the osculating plane of $\mathcal{C}$ at $\boldsymbol{r}(t)$.

(c) Find the curvature $\kappa(t)$ of $\mathcal{C}$ at $\boldsymbol{r}(t)$.

Solution:

$$\gamma'(t) = ((1+t)^{\frac{1}{2}}, -(1-t)^{\frac{1}{2}}, a)$$
$$|\mathbf{T}'(t)| = \sqrt{a^2+2}$$
$$s(b) = \int_0^b |\mathbf{T}'(t)|dt = b\sqrt{a^2+2}\ldots\ldots(3pts)$$
$$\mathbf{T}(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{\sqrt{a^2+2}}((1+t)^{\frac{1}{2}}, -(1-t)^{\frac{1}{2}}, a)\ldots\ldots(3pts)$$
$$\mathbf{T}'(t) = \frac{1}{\sqrt{a^2+2}}(\frac{1}{2}(1+t)^{\frac{-1}{2}}, \frac{1}{2}(1-t)^{\frac{-1}{2}}, 0) = \frac{1}{2\sqrt{a^2+2}}(1+t)^{\frac{-1}{2}}, (1-t)^{\frac{-1}{2}}, 0)$$
$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{2(a^2+2)(1-t^2)}}$$
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}}((1-t)^{\frac{1}{2}}, (1+t)^{\frac{1}{2}}, 0)\ldots\ldots(3pts)$$
$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{2(a^2+2)}}(-a(1+t)^{\frac{1}{2}}, a(1-t)^{\frac{1}{2}}, 2)$$
$$\text{Osculating plane at}\gamma(t) : (x - \frac{2}{3}(1+t)^{\frac{2}{3}}, y - \frac{2}{3}(1-t)^{\frac{2}{3}}, z - at)\cdot B(t) = 0\ldots\ldots(3pts)$$
$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\gamma'(t)|} = \frac{1}{\sqrt{2}(a^2+2)\sqrt{1-t^2}},\quad t \in (-1,1)\ldots\ldots(3pts)$$

4. (15 points) Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

(b) Calculate $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(c) Are f_x and f_y continuous at $(0,0)$?

(d) Determine $f_{xy}(0,0)$ and $f_{yx}(0,0)$ if they exist. If they do not exist, explain why.

(e) Is $f(x,y)$ differentiable at $(0,0)$?

Solution:

(a) by definition ,

$\frac{\partial f}{\partial x}|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$

similaritly ,

$\frac{\partial f}{\partial y}|_{(0,0)} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$ (each 1 pts.)

(b) when f on $(x,y) \neq (0,0)$

$\frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2)-x^2y(2x)}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y^2)-x^2y(2y)}{(x^2+y^2)^2} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$ (each 2pts.)

(c) observe f along $y = mx$, m arbitrary

$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,mx)\to(0,0)} \frac{2x(mx)^3}{(x^2+(mx)^2)^2} = \frac{2m^3}{(1+m^2)^2}$

$\neq f_x(0,0) = 0$

$\lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,mx)\to(0,0)} \frac{x^2(x^2-(mx)^2)}{(x^2+(mx)^2)^2} = \frac{1-m^2}{(1+m^2)^2}$

$\neq f_y(0,0) = 0$

$\Rightarrow$ limit doesn't exist at $(0,0)$

$\Rightarrow f_x, f_y$ not conti. at $(0,0)$ (3 pts.)

(d) $f_{xy} = \frac{\partial f_x}{\partial y}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = 0$

$f_{yx} = \frac{\partial f_y}{\partial x}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \longrightarrow \infty$ (3 pts.)

(e) sol.1 If f diff. at $(0,0)$, then $\lim_{\sqrt{h^2+k^2}\to 0} \frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} = 0$ (1 pts)

$\Rightarrow \lim_{\sqrt{h^2+k^2}\to 0} \frac{h^2k}{(h^2+k^2)^{\frac{3}{2}}} = 0$

But along $h = k$

$\lim_{\sqrt{h^2+k^2}\to 0} \frac{h^2k}{(h^2+k^2)^{\frac{3}{2}}} = \lim_{h\to 0} \frac{h^3}{2^{\frac{3}{2}}h^3} = 2^{\frac{-3}{2}} \neq 0$ so f not diff. at$(0,0)$ (3 pts.)

sol.2 set $\overrightarrow{u} = (\frac{1}{\sqrt{m^2+1}}, \frac{m}{\sqrt{m^2+1}})$

$f_u(0,0) = \lim_{h\to 0} \frac{f(\frac{h}{\sqrt{m^2+1}}, \frac{mh}{\sqrt{m^2+1}}) - f(0,0)}{h} = \frac{m}{(1+m^2)^{\frac{3}{2}}}$

But $\nabla f(0,0)\cdot\overrightarrow{u} =< f_x(0,0), f_y(0,0) > \cdot\overrightarrow{u} = 0$

a contradiction , so f not diff. at $(0,0)$ (3 pts.)

5. (15 points) Let $f(x,y,z) = e^{xy}\ln z$. Find the directional derivatives of f at $P(1,0,e)$ in the following directions.

(a) In the direction in which f increases most rapidly at P.

(b) In the directions parallel to the line in which the planes $x+y-z=2$ and $4x-y-z=1$ intersect.

(c) In the direction of increasing t along the path

$$\boldsymbol{r}(t) = \sqrt{1+t^2}\boldsymbol{i} + \tan t\boldsymbol{j} + e^{2t+1}\boldsymbol{k}$$

Solution:

$$f(x,y,z) = e^{xy}\ln z$$
$$\Rightarrow \nabla f(x,y,z) = (ye^{xy}\ln z)\ \mathbf{i} + (xe^{xy}\ln z)\ \mathbf{j} + (\frac{1}{z}e^{xy})\ \mathbf{k} \quad \textbf{(2 pts)}$$
$$\Rightarrow \nabla f(1,0,e) = \mathbf{j} + \frac{1}{e}\ \mathbf{k} \quad \textbf{(1 pt)}$$

(**Get the correct expression of $\nabla f(x,y,z)$ but the wrong value of $\nabla f(1,0,e)$: 2-point deduction for whole question.**)

(a)

The desired directional derivative is $\|\nabla f(1,0,e)\| = \frac{\sqrt{1+e^2}}{e}$. **(4 pts)**

(b)
The directions of this line are $\mathbf{v} = (1,1,-1)\times(4,-1,-1) = (-2,-3,-5)$ and $-\mathbf{v} = (2,3,5)$.
$\Rightarrow$ The unit vectors are $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{38}}(-2,-3,-5)$ and $-\mathbf{u} = \frac{1}{\sqrt{38}}(2,3,5)$. **(2 pts)**
$\Rightarrow$ The directional derivative in $\mathbf{u}$ is $\nabla f(1,0,e)\cdot\mathbf{u} = \frac{-1}{\sqrt{38}}(3+5e^{-1})$, and directional derivative the in $-\mathbf{u}$ is $\frac{1}{\sqrt{38}}(3+5e^{-1})$. **(2 pts)**
(**If you only write one of the two derivatives, you get at most 3 pts.**)

(c)
$\mathbf{r}'(t) = \frac{t}{\sqrt{1+t^2}}\ \mathbf{i} + \sec^2 t\ \mathbf{j} + 2e^{2t+1}\ \mathbf{k}$.
$\Rightarrow \mathbf{r}'(0) = \mathbf{j} + 2e\ \mathbf{k}$. **(2 pts)**
$\Rightarrow$ The desired directional derivative is $\nabla f(\mathbf{r}(0))\cdot\frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{3}{\sqrt{1+4e^2}}$. **(2 pts)**

(**Calculation error: 1-point deduction for each error.**)
(**Correct formula but with wrong answer form: 1-point deduction for each error.**)
(**Did not use unit vectors: 1-point deduction for each error.**)

6. (15 points) Suppose $f(x,y) = x^2 + cxy + 2y^2$ where c is a constant.
   (a) Find all values of c such that $(0,0)$ is a stationary point of f.
   (b) Find all values of c such that $(0,0)$ is a saddle point of f.
   (c) Find all values of c such that f has a local minimum at $(0,0)$.
   (d) Find all values of c and all $(x_0,y_0)\neq(0,0)$ such that f has a local minimum at (x_0,y_0).

Solution:
(a) (3 %)
solution:
For $\nabla f = (2x+cy)\hat{\imath} + (cx+4y)\hat{\jmath}$, we have a point (x,y) is a stationary point if $\nabla f(x,y) = 0$, that is $2x+cy = 0$ and $cx+4y = 0$. So, for $(0,0)$ to be a stationary point of f, it is clear that c can be any real number, i.e. $c\in\mathbb{R}$.

(b) (4 %)
solution:
For $\nabla f = (2x+cy)\hat{\imath} + (cx+4y)\hat{\jmath} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$, we have $\frac{\partial^2 f}{\partial x^2}(x,y) = 2$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 4$, and $\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y) = c$ for all $(x,y)\in\mathbb{R}^2$. Thus, at $(0,0)$, $A = \frac{\partial^2 f}{\partial x^2}(0,0) = 2$,$C = \frac{\partial^2 f}{\partial y^2}(0,0) = 4$, and $B = \frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial^2 f}{\partial y\partial x}(0,0) = c$.
The discriminant is $D = AC - B^2 = 8 - c^2$.
By second partials test, for $(0,0)$ to be a saddle point, we must have $D < 0$, that is $8 - c^2 < 0$, so $c > 2\sqrt{2}$ and $c < -2\sqrt{2}$.

If you do this problem only until here, you can get 4 points ,but the check for the case $D = 0$ will be 2 points in next problem (c).

When $D = 0$, we have $c = \pm 2\sqrt{2}$, so if $c = 2\sqrt{2}$, we have $f(x, y) = x^2 + 2\sqrt{2}xy + 2y^2 = (x + \sqrt{2}y)^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, thus $(0, 0)$ is a local minimum if $c = 2\sqrt{2}$. Similarly, when $c = -2\sqrt{2}$, we have $f(x, y) = x^2 - 2\sqrt{2}xy + 2y^2 = (x - \sqrt{2}y)^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, thus $(0, 0)$ is a local minimum if $c = -2\sqrt{2}$.

Therefore, the point $(0, 0)$ si a saddle point only when $c > 2\sqrt{2}$ and $c < -2\sqrt{2}$.
(c) (4 %)
solution:
The discriminant is $D = AC - B^2 = 8 - c^2$.
By second partials test, for $(0, 0)$ to be a local minimum, we must have $D > 0$ and $A > 0$, but $A = \dfrac{\partial^2 f}{\partial x^2}(0, 0) = 2 > 0$, which is clear. So we only need to consider $D > 0$, that is $8 - c^2 > 0$, so $-2\sqrt{2} < c < 2\sqrt{2}$.
If you do this problem only until here, you can get 2 points.
By the argument in the problem (b), we know that when $c = \pm 2\sqrt{2}$, $(0, 0)$ is a locl minimum. Thus $(0, 0)$ si a locl minimum only when $-2\sqrt{2} \leq c \leq 2\sqrt{2}$
(d) (4%)
solution:
Note that if a point (x_0, y_0) is a locl minimum of f, we must have the point (x_0, y_0) satisfies $\nabla f(x_0.y_0) = 0$, that is $2x_0 + cy_0 = 0$ and $cx_0 + 4y_0 = 0$. But to have the point $(x_0, y_0) \neq (0, 0)$, we need the above system of equations ($2x_0 + cy_0 = 0$ & $cx_0 + 4y_0 = 0$) have solutions other than $(0, 0)$ this is equivalent to $8 - c^2 = 0$ (which is the determinant of the matrix of the coefficints of the above system of equations).
So $c = \pm 2\sqrt{2}$, when $c = 2\sqrt{2}$, we have $f(x, y) = x^2 + 2\sqrt{2}xy + 2y^2 = (x + \sqrt{2}y)^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, thus we have all the points on the line $x + 2\sqrt{2}y = 0$ are local minimum of f.
Similarly, when $c = -2\sqrt{2}$,we have $f(x, y) = x^2 - 2\sqrt{2}xy + 2y^2 = (x - \sqrt{2}y)^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, thus we have all the points on the line $x - 2\sqrt{2}y = 0$ are local minimum of f.
Therefore, the value of c are $\pm 2\sqrt{2}$, and the corresponding $(x_0, y_0) \neq (0, 0)$ are the set $\{(x, y) \neq (0, 0) : x + \sqrt{2}y = 0\}$ and $\{(x, y) \neq (0, 0) : x - \sqrt{2}y = 0\}$, respectively.

7. (15 points) A rectangular box has three of its faces on the coordinate planes and one vertex in the first octant on the paraboloid $z = 4 - 5x^2 - 6y^2$. Determine the maximum volume of the box.

Solution:
We want to find the maximum of xyz with side condition $z = 4 - 5x^2 - 6y^2$. So putting $f(x, y, z) = xyz$ and $g(x, y, z) = 5x^2 + 6y^2 + z$, and using Lagrang's method by setting $\nabla f = \lambda \nabla g$, we have

$$\begin{cases} yz = 10\lambda x \\ xz = 12\lambda y \\ xy = \lambda. \end{cases}$$

Substituting $xy = \lambda$ to the first and the second equation, we have

$$\begin{cases} yz = 10x^2 y \\ xz = 12\lambda xy^2. \end{cases}$$

Hence, we get $z = 10x^2 = 12y^2$, since $z = 4 - 5x^2 - 6y^2$, we get $x^2 = \dfrac{1}{5}$, $y^2 = \dfrac{1}{6}$, and $z = 2$ (also we get $\lambda = \dfrac{1}{\sqrt{30}}$) when xyz attains extrema. We then deduce the maximum should be $\sqrt{\dfrac{2}{15}}$.

評分標準：

(a) 算出∇f及∇g，並列出$\nabla f = \lambda \nabla g$以明示使用Lagrange方法，得2分
(b) 滿足前述條件且列出$yz = 10\lambda x$, $xz = 12\lambda y$, $xy = \lambda$之明顯的等價敘述，得3分
(c) 滿足前述條件且列出$10x^2 = 12y^2$之明顯的等價敘述，得3分
(d) 滿足前述條件且列出$z = 10x^2 = 12y^2$之明顯的等價敘述，得2分
(e) 滿足前述條件且得到正確答案，得5分。但若滿足前述條件且得到達最大值之座標、比值或λ卻計算出錯誤答案，得3分。
(f) 使用其他方法（例如：第二偏導數判定法、算幾不等式等等）斟酌給分。