1. (10%) Evaluate $\int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6+y^2}} dydx$.

Solution:

Interchange the order of the iterated integral, we have

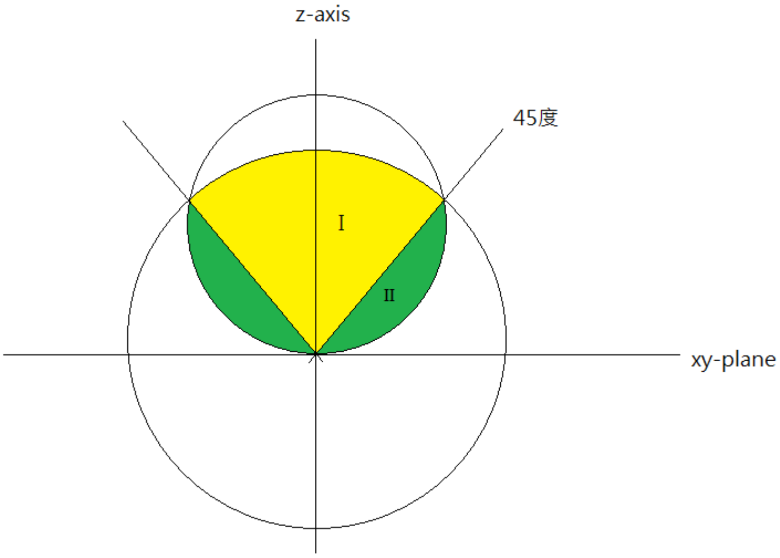
$$\begin{aligned}
\int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6+y^2}}\, dydx &= \int_0^8 \int_0^{\sqrt[3]{y}} \frac{x^5}{\sqrt{x^6+y^2}}\, dxdy \quad (3\%) \\
&= \int_0^8 \frac{1}{6} 2\sqrt{x^6+y^2}\Big|_0^{\sqrt[3]{y}}\, dy \quad (3\%) \\
&= \frac{1}{3}\int_0^8 (\sqrt{2}-1) y\, dy \quad (2\%) \\
&= \frac{1}{3}(\sqrt{2}-1)\frac{1}{2}y^2\Big|_0^8 \quad (1\%) \\
&= \frac{32}{3}(\sqrt{2}-1) \quad (1\%)
\end{aligned}$$

2. (15%) Find the volume of the solid common to the balls $\rho \le 2\sqrt{2}\cos\phi$ and $\rho \le 2$.

Solution:

Method 1.

Sphere $\rho = 2$ is equivalent to $x^2+y^2+z^2 = 4$; sphere $\rho = 2\sqrt{2}\cos\phi$ is equivalent to $x^2+y^2+(z-\sqrt{2})^2 = 2$. These two spheres intersect at $\phi = \pi/4$.



Therefore,

$$\text{Volume} = \underbrace{\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\phi \int_0^{2} \rho^2 \sin\phi \, d\rho}_{\text{I}} + \underbrace{\int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi \int_0^{2\sqrt{2}\cos\phi} \rho^2 \sin\phi \, d\rho}_{\text{II}}$$

$$= 2\pi(-\cos\phi)\Big|_0^{\frac{\pi}{4}} \cdot \left(\frac{\rho^3}{3}\right)\Big|_0^{2} + 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \cdot \frac{1}{3}(2\sqrt{2}\cos\phi)^3 \, d\phi$$

$$= \underline{\frac{2\pi}{3}(8-4\sqrt{2})} + \frac{32\sqrt{2}\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \cos^3\phi \, d\phi$$

$$= \frac{2\pi}{3}(8-4\sqrt{2}) + \underline{\frac{2\pi}{3}}$$

$$= \frac{16}{3}\pi - 2\sqrt{2}\pi.$$

Note:

$\boxed{\theta : 0 \sim 2\pi, \text{ (2 points)}}$, Jacobi factor $\boxed{\rho^2 \sin\phi, \text{ (5 points)}}$.

For part I,, $\boxed{\phi : 0 \sim \pi/4, \text{ (2 points)}}$, $\boxed{\rho : 0 \sim 2, \text{ (1 point)}}$, answer $\boxed{\frac{2\pi}{3}(8-4\sqrt{2}), \text{ (1 point)}}$

For part II, $\boxed{\phi : \pi/4 \sim \pi/2, \text{ (2 points)}}$, $\boxed{\rho : 0 \sim 2\sqrt{2}\cos\phi, \text{ (1 point)}}$, answer $\boxed{\frac{2\pi}{3}, \text{ (1 point)}}$

Method 2.

$$\text{Volume} = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} dr \int_{\sqrt{2}-\sqrt{2-r^2}}^{\sqrt{4-r^2}} r \, dz$$

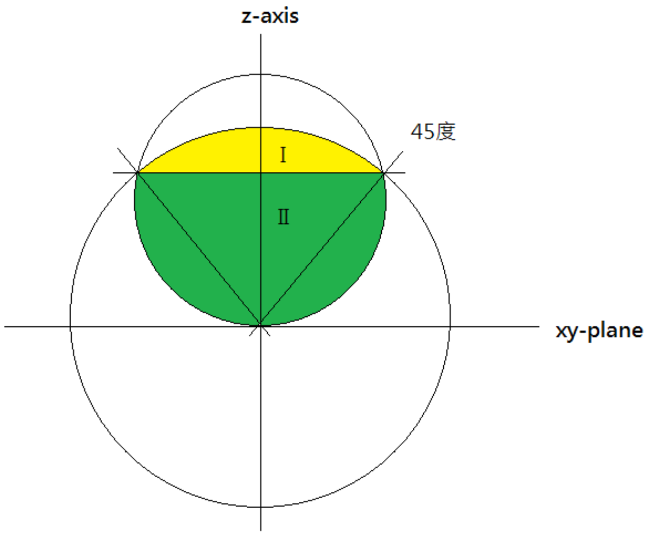
$$= 2\pi \int_0^{\sqrt{2}} r(\sqrt{4-r^2} + \sqrt{2-r^2} - \sqrt{2}) \, dr$$

$$= \frac{16}{3}\pi - 2\sqrt{2}\pi.$$

Note:
$\boxed{\theta : 0 \sim 2\pi, \text{ (2 points)}}$, $\boxed{z : \sqrt{2} - \sqrt{2 - r^2} \sim \sqrt{4 - r^2}, \text{ (3 points)}}$, $\boxed{r : 0 \sim \sqrt{2}, \text{ (3 points)}}$,

Jacobi factor $\boxed{r, \text{ (5 points)}}$, answer $\boxed{\frac{2\pi}{3}(8 - 3\sqrt{2}), \text{ (2 points)}}$.

Method 3.



$$\begin{aligned} \text{Volume} &= \underbrace{\int_{\sqrt{2}}^{2} \pi(4 - z^2)\, dz}_{\text{I}} + \underbrace{\frac{4}{3}\pi(\sqrt{2})^3 \cdot \frac{1}{2}}_{\text{II}} \\ &= \frac{16}{3}\pi - 2\sqrt{2}\pi. \end{aligned}$$

3. (15%) Evaluate the integral $\iint\limits_{\Omega} \sin(3x^2 - 2xy + 3y^2)dxdy$, where Ω is the ellipse $3x^2 - 2xy + 3y^2 \le 2$. You may try the change of variables $x = u + kv$, $y = u - kv$ for some constant k.

Solution:

Follow the hint, set $x = u + kv, y = u - kv$. Put in equation of the ellipse: **(3 pts)**

$$3x^2 - 2xy + 3y^2 = 4u^2 + 8k^2v^2 \le 2$$

We can choose $k = \frac{1}{\sqrt{2}}$, then the equation becomes $4u^2 + 4v^2 \le 2$. Calculate the Jacobian **(value of J 2 pts , absolute value 2 pts)**:

$$|J(u,v)| = |\frac{\partial(x,y)}{\partial(u,v)}| = | \begin{vmatrix} 1 & k \\ 1 & -k \end{vmatrix} | = |-2k| = \sqrt{2}$$

Then

$$\begin{aligned}
&\iint\limits_{\Omega} \sin(3x^2 - 2xy + 3^2)\, dxdy \\
=&\iint\limits_{u^2+v^2\le\frac{1}{2}} \sin 4(u^2+v^2)\sqrt{2}\, dudv \quad \textbf{(integrand 1 pt, domain 2 pts)} \\
=&\sqrt{2}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \sin 4r^2\, rdrd\theta \quad \textbf{(Jacobian of polar coordinates 3 pts)} \\
=&2\sqrt{2}\pi\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{8}\sin 4r^2\, d(4r^2) \\
=&\frac{\sqrt{2}\pi}{4}(1-\cos 2) \quad \textbf{(2 pts. If make a slight mistake, get 1 pt)}
\end{aligned}$$

Scoring to steps of this problem:

1. Change of variable in u, v: get 2 pt if the relation is correct.
2. Jacobian of your variable: get 2 pts for the value and 2 pts for absolute value.
3. Write down the correct integrand for your new variables: get 1 pt.
4. Your integral domain is correct: get 2 pt.
5. Change u, v into polar coordinates. If the Jacobian is correct: get 3 pts.
6. Your result fits the correct answer: get 2pts, and get 1 pt if you just make a slight mistake.

4. (15%) For $y > 0$, let

$$\mathbf{F}(x,y,z) = (e^{-x}\ln y - z)\mathbf{i} + \left(2yz - e^{-x}/y\right)\mathbf{j} + (y^2 - x)\mathbf{k} \text{ and}$$
$$\mathbf{G}(x,y,z) = e^{-x}\ln y\,\mathbf{i} + \left(2yz - e^{-x}/y\right)\mathbf{j} - x\,\mathbf{k}.$$

(a) Show that the vector function $\mathbf{F}$ is a gradient on $\{(x,y,z)|\ y > 0\}$ by finding an f such that $\nabla f = \mathbf{F}$.

(b) Evaluate the line integral $\int_C \mathbf{G}(\mathbf{r}) \cdot \mathbf{dr}$, where C is the curve given by $\mathbf{r}(u) = (1+u^2)\mathbf{i} + e^u\mathbf{j} + (1+u)\mathbf{k}$, $u \in [0,1]$.

Solution:

(a) $f = -e^{-x}\ln y + y^2 z - xz + c$, where $c \in \mathbf{R}$.(5 points)

(b) $G = F + z\mathbf{i} - y^2\mathbf{k}$, and $r(0) = (1,1,1),\ r(1) = (2,e,2)$.

Then

$$\int_C G(r) \cdot dr = \int_C F(r) \cdot dr + \int_C zdx - y^2 dz$$
$$= f(2,e,2) - f(1,1,1) + \int_0^1 [(1+u)2u - e^{2u}]du (\text{4points})$$
$$= -e^{-2} + 2e^2 - 4 + (\frac{2}{3}u^3 + u^2 - \frac{1}{2}e^{2u})|_0^1 (\text{4 points})$$
$$= \frac{3}{2}e^2 - e^{-2} - \frac{11}{6} (\text{2points})$$

5. (15%) Let C be a piecewise-smooth Jordan curve that does not pass through the origin. Evaluate $\oint_C \frac{-y^5}{(x^2+y^2)^3}dx + \frac{xy^4}{(x^2+y^2)^3}dy$ for the following two cases, where C is traversed in the counter-clockwise direction.

(a) C does not enclose the origin.

(b) C does enclose the origin.

Solution:

(a)

Let Ω be the region enclosed by C. Since Ω does not enclose the origin, the functions

$$P(x,y) = \frac{-y^5}{(x^2+y^2)^3}$$

and

$$Q(x,y) = \frac{xy^4}{(x^2+y^2)^3}$$

are well-defined and differentiable in Ω. We have

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y) = \frac{y^6 - 5x^2y^4}{(x^2+y^2)^4}$$

in Ω. Therefore, by Green's theorem,

$$\oint_C Pdx + Qdy = \int\int_\Omega \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dxdy = 0.$$

Grading Policy:

Application of Green's theorem: 2%

Correct calculation of partial derivatives: 2%

Correct answer 3%

(b)

Let C_r be the curve $\theta \mapsto (r\cos\theta, r\sin\theta), \theta \in [0, 2\pi]$, where r is small enough such that C_r lies in the interior of the region bounded by C. Let Ω be the region bounded by C and C_r. By Green's theorem, we have

$$\oint_C Pdx + Qdy - \oint_{C_r} Pdx + Qdy = \int\int_\Omega \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dxdy,$$

where $P(x,y)$ and $Q(x,y)$ are defined as in (a). Since Ω does not contain the origin, the right hand side of the above equation is 0. Thus

$$\begin{aligned}
\oint_C Pdx + Qdy &= \oint_{C_r} Pdx + Qdy \\
&= \int_0^{2\pi} \left(\frac{-r^5\sin^5\theta}{r^6}(-r\sin\theta) + \frac{r^5\cos\theta\sin^4\theta}{r^6}(r\cos\theta)\right) d\theta \\
&= \int_0^{2\pi} \sin^4\theta d\theta \\
&= \frac{3\pi}{4}.
\end{aligned}$$

Grading Policy:

Valid application of Green's theorem: 3%

Correctly transforming the line integral to the ordinary integral: 2%

Correct answer: 3%

6. (15%) Let S be the triangular region with vertices $(0,0,0)$, $(a,0,0)$, and (a,a,a), $a>0$, with upward unit normal $\mathbf{n}$, and C be the positively oriented boundary of S. Let

$$\mathbf{F} = \left(y - z\cos(x^2)\right)\mathbf{i} + \left(2x - \sin(z^2)\right)\mathbf{j} + \left(3z - \tan(y^2)\right)\mathbf{k}.$$

(a) Find a parametrization of S and find the upward unit normal $\mathbf{n}$. (Hint. Consider the projection of S to xy-plane.)

(b) Evaluate $\nabla \times \mathbf{F}$.

(c) Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Solution:

(a) i) $S : \{(x,y,y) |\ 0 \le y \le x \le a\ \}$ (2 points)

ii) $\mathbf{n} = \dfrac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{0}, -\mathbf{1}, \mathbf{1})$ (2 points)

(b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$ (2 points)

$= \left[2z\cos z^2 - 2y\sec^2 y^2\right]\mathbf{i} - \cos x^2\ \mathbf{j} + \mathbf{k}$ (2 points)

(c) $\oint_c \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot \mathbf{n}\ d\sigma = \int_0^a \int_0^x \frac{1}{\sqrt{2}}(1+\cos x^2)\cdot |\mathbf{r_x} \times \mathbf{r_y}| dy dx$ (4 points)

$= \int_0^a x(1+\cos x^2)dx = \dfrac{x^2 + \sin x^2}{2}|_0^a = \dfrac{1}{2}(a^2 + \sin a^2)$ (3 points)

7. (15%) Let S_1 be the surface $\{(x,y,z)|\ z=x^2+y^2,\ z\le y\}$, S_2 be the surface $\{(x,y,z)|\ z=y,\ x^2+y^2\le z\}$, and $\mathbf{V}(x,y,z)=-y\mathbf{i}+x\mathbf{j}+z\mathbf{k}$.

(a) Compute directly the downward flux of $\mathbf{V}$ across S_1.

(b) Use the divergence theorem to compute the upward flux of $\mathbf{V}$ across S_2.

Solution:

7.(a)

$$\begin{aligned}
S_1 : f_1(x,y) &= (x,y,x^2+y^2), \quad x^2+y^2\le y\\
\frac{\partial f_1}{\partial x} =&(1,0,2x)\\
\frac{\partial f_1}{\partial y} =&(0,1,2y)\\
\frac{\partial f_1}{\partial x}\times\frac{\partial f_1}{\partial y} =&(-2x,-2y,1)\\
\mathbf{d}\,\text{area} =&(2x,2y,-1)\,dx\,dy \quad \text{(since the direction is downward)} \quad \text{(1 pt)}\\
\mathbf{V} =&(-y,x,z)
\end{aligned}$$

Let $x=r\cos\theta, y=r\sin\theta$. Then $r^2\le r\sin\theta \Rightarrow 0\le r\le\sin\theta, 0\le\theta\le\pi$.

$$\begin{aligned}
\iint_{S_1}\mathbf{V}\cdot\mathbf{d}\,\text{area} &= \iint_{S_1} -2xy+2xy-(x^2+y^2)\,dx\,dy \quad \text{(1 pt)}\\
&= \int_{\theta=0}^{\pi}\int_{r=0}^{\sin\theta} -r^2\,r\,dr\,d\theta \quad \text{(3 pts)}\\
&= \frac{-1}{4}\int_0^{\pi}\sin^4\theta\,d\theta = \frac{-1}{4}\frac{3}{4}\frac{1}{2}\pi\\
&= \frac{-3\pi}{32} \quad \text{(2 pts)}
\end{aligned}$$

(b)

Let $x=r\cos\theta, y=r\sin\theta$.

$$\Omega : x^2+y^2\le z\le y \Rightarrow \begin{cases} r^2\le z\le y\\ r^2\le r\sin\theta\end{cases} \Rightarrow \begin{cases} r^2\le z\le y\\ 0\le r\le\sin\theta\\ 0\le\theta\le\pi\end{cases}$$

By divergent theorem,

$$\iiint_{\Omega}\nabla\cdot\mathbf{V}\,d\,\text{volumn} = \iint_{\partial\Omega=S_1+S_2}\mathbf{V}\cdot\mathbf{d}\,\text{area} = \iint_{S_1}\mathbf{V}\cdot\mathbf{d}\,\text{area}+\iint_{S_2}\mathbf{V}\cdot\mathbf{d}\,\text{area} \quad \text{(1 pt)}.$$

$$\begin{aligned}
\iiint_{\Omega}\nabla\cdot\mathbf{V}\,d\,\text{volumn} &= \int_{\theta=0}^{\pi}\int_{r=0}^{\sin\theta}\int_{z=r^2}^{r\sin\theta} 1\ r\,dz\,dr\,d\theta \quad \text{(3 pts)}\\
&= \int_{\theta=0}^{\pi}\int_{r=0}^{\sin\theta}(r\sin\theta-r^2)\ r\,dr\,d\theta\\
&= \int_{\theta=0}^{\pi}\sin\theta\frac{r^3}{3}\Big|_0^{\sin\theta}-\frac{r^4}{4}\Big|_0^{\sin\theta}\,d\theta\\
&= \int_0^{\pi}\frac{1}{3}\sin^4\theta-\frac{1}{4}\sin^4\theta\,d\theta\\
&= \frac{1}{12}\left(\frac{3}{4}\frac{1}{2}\pi\right) = \frac{\pi}{32} \quad \text{(2 pts)}
\end{aligned}$$

$$\Rightarrow \iint_{S_2}\mathbf{V}\cdot\mathbf{d}\,\text{area} = \frac{\pi}{32}-\left(\frac{-3\pi}{32}\right) = \frac{\pi}{8} \quad \text{(2 pts)}$$