1011微甲07-11班期中考解答和評分標準

1. (15%) Find $\lim_{x\to 0^-}$ and $\lim_{x\to 0^+}$ of the following functions:

$$(a)\ \frac{\sin(|x|)}{x},\quad (b)\ \frac{\cos x - 1}{\sin(x\sin x)},\quad (c)\ \frac{\cos(\sin x)-1}{\tan^2 x}$$

Solution:

(a) 5 points

$$\lim_{x\to 0^-}\frac{\sin|x|}{x} = \lim_{x\to 0^-}\frac{\sin(-x)}{x} = \lim_{x\to 0^-} -\frac{\sin x}{x} = -1$$

$$\lim_{x\to 0^+}\frac{\sin|x|}{x} = \lim_{x\to 0^+}\frac{\sin x}{x} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1$$

(b) 5 points

$$\lim_{x\to 0}\frac{\cos x - 1}{\sin(x\sin x)} = \lim_{x\to 0}\frac{\cos x - 1}{\sin(x\sin x)}\left(\frac{\cos x + 1}{\cos x + 1}\right) =$$

$$\lim_{x\to 0}\frac{\cos^2 x - 1}{[\sin(x\sin x)](\cos x + 1)} = \lim_{x\to 0}\frac{-\sin^2 x}{[\sin(x\sin x)](\cos x + 1)}$$

$$= \lim_{x\to 0}\frac{-\sin^2 x}{[\sin(x\sin x)](\cos x + 1)}\frac{x\sin x}{x\sin x} =$$

$$\lim_{x\to 0}[-\sin(x)/x]\lim_{x\to 0}[(x\sin(x)/\sin(x\sin(x))]\lim_{x\to 0}\left(\frac{1}{\cos x + 1}\right) = \frac{-1}{2}$$

(c) 5 points

$$\lim_{x\to 0}\frac{\cos(\sin x) - 1}{\tan^2 x} = \lim_{x\to 0}\frac{\cos^2(\sin x) - 1}{\tan^2 x(\cos(\sin x) + 1)} = \lim_{x\to 0}\frac{-\sin^2(\sin x)\cos^2 x}{\sin^2 x(\cos(\sin x) + 1)}$$

$$= \lim_{x\to 0}[-\sin^2(\sin x)/\sin^2 x]\lim_{x\to 0}\frac{\cos^2 x}{\cos(\sin x) + 1} = \frac{-1}{2}$$

2. (10%) Show that $|\tan\frac{x}{2} - \tan\frac{y}{2}| \geq \frac{|x-y|}{2}$ for $x, y \in (-\pi, \pi)$.

Solution:

Let $x, y \in (-\pi, \pi)$, W.L.O.G, set $x < y$.

Let $f(t) = \tan\frac{t}{2}$, then f is continuous on $[x, y]$ and differentiable on (x, y). (2 point)

By Mean Value Theorem, there is a number c between x and y such that

$\frac{f(x) - f(y)}{x - y} = f'(c)$ (2 point)

Since $f'(c) = \frac{1}{2}\sec^2\frac{c}{2}$ (2 point)

$\Rightarrow \frac{|\tan\frac{x}{2} - \tan\frac{y}{2}|}{|x - y|} = |\frac{1}{2}\sec^2\frac{c}{2}|$

$\Rightarrow |\tan\frac{x}{2} - \tan\frac{y}{2}| = |\frac{1}{2}\sec^2\frac{c}{2}||x - y|$ (2 point)

We know that $|\frac{1}{2}\sec^2\frac{c}{2}| \geq \frac{1}{2}$, (1 point)

$\Rightarrow |\tan\frac{x}{2} - \tan\frac{y}{2}| \geq \frac{|x - y|}{2}$ (1 point)

3. (10%) A rhombus (菱形) has sides 10in. long. Two of its opposite vertices are pulled apart at a rate of 2 in. per second. How fast is the area changing when the vertices being pulled are 16 in apart?

Solution:

Let the distance between the two pulled vertices is x in., and the length of another diagonal is y in.. The change rate of x is $\frac{dx}{dt} = 2$.

By Pythagorean theorem, $(\frac{x}{2})^2 + (\frac{y}{2})^2 = 10^2 = 100 \Longrightarrow x^2 + y^2 = 400 \Longrightarrow y = \sqrt{400 - x^2}$

So the area A of the rhombus is $\frac{xy}{2} = \frac{1}{2}x\sqrt{400 - x^2}$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \left(\frac{1}{2}\sqrt{400 - x^2} - \frac{1}{2} \cdot \frac{x^2}{\sqrt{400 - x^2}}\right) \times 2 = \sqrt{400 - x^2} - \cdot \frac{x^2}{\sqrt{400 - x^2}}$

When $x = 16$, $\left.\frac{dA}{dt}\right|_{x=16} = 12 - \frac{256}{12} = -\frac{28}{3}$ (in^2/sec)

評分標準如下:

寫出長度或者角度之間的關係 (2分)

寫出面積與所設變數之間的關係式 (2分)

將面積對變數作微分 (4分)

代入欲求取之值 (2分)

其餘錯誤酌量扣分。

4. (10%) Let $f(x) = \frac{1 + \cos x}{1 + \sin x}$. Use a differential to estimate $f(44°)$.

Solution:

$f(x + h) \simeq f(x) + f'(x) \cdot h$

$f'(x) = \frac{-\sin x(1 + \sin x) - \cos x(1 + \cos x)}{(1 + \sin x)^2} = \frac{-\sin x - \cos x - 1}{(1 + \sin x)^2}$ (4 points)

$f(44°) = f(45° + (-1°)) \simeq f(45°) + f'(45°) \cdot (-1°)$ (2 points)

$= f(\frac{\pi}{4}) + f'(\frac{\pi}{4}) \cdot \frac{-\pi}{180}$ (2 points)

$= 1 + (2 - 2\sqrt{2}) \cdot \frac{-\pi}{180}$

$= 1 + \frac{(\sqrt{2} - 1)\pi}{90}$. (1 point)

其中:

$f(\frac{\pi}{4}) = \frac{1 + \cos\frac{\pi}{4}}{1 + \sin\frac{\pi}{4}} = \frac{1 + \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = 1$

$f'(\frac{\pi}{4}) = \frac{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 1}{(1 + \frac{\sqrt{2}}{2})^2} = \frac{-\sqrt{2} - 1}{\frac{3}{2} + \sqrt{2}} = \frac{-2\sqrt{2} - 2}{3 + 2\sqrt{2}} \cdot \frac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}}$

$= (-2\sqrt{2} - 2)(3 - 2\sqrt{2})$

$= 2 - 2\sqrt{2}$. (1 points)

5. (25%) Let $f(x) = \frac{(x + 1)^2}{x^2 + 1}$.

(a) (5%) Find f' and f''.

(b) (10%) Find the intervals on which f increases and the intervals on which f decreases. Indicate local extreme values and absolute extreme values.

(c) (5%) Find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down. Indicate points of inflection.

(d) (5%) Find vertical and horizontal asymptotes if any. Sketch the graph of f.

Solution:

(a)

$f(x) = (x^2 + 1)^{-1}(x + 1)^2$

$f'(x) = -(x^2+1)^{-2}(2x)(x+1)^2 + 2(x^2+1)^{-1}(x+1) = \dfrac{-2(x+1)(x-1)}{(x^2+1)^2}$ (3%)

$f''(x) = -2[-2(x^2+1)^{-3}(2x)(x+1)(x-1) + (x^2+1)^{-2}(2x)] = \dfrac{4x(x^2-3)}{(x^2+1)^3}$ (2%)

(b)

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	−	−	−	0	+	+	+	0	−	−	−
$f''(x)$	−	0	+	+	+	0	−	−	−	0	+
$f(x)$	╮		╰	0	╯	1	╭	2	╮		╰

From the above chart we know that

(1) f is increasing on $[-1, 1]$ (2%)

(2) f is decreasing on $(-\infty, -1]$ and $[1, \infty]$ (2%)

(3) Local minimum: $f(-1) = 0$ (2%); local maximum: f(1) = 2 (2%)

(4) Absolute minimum: $f(-1) = 0$; absolute maximum: f(1) = 2 (1%) since $\lim_{x\to\pm\infty} f(x) = 1$ and $f(x)$ is finite for any real x (reasoning 1%).

(c) $f''(x)$ is 0 and changes its sign at $x = 0, \pm\sqrt{3}$

$\Rightarrow$ inflection points: $x = 0, \pm\sqrt{3}$ (3%).

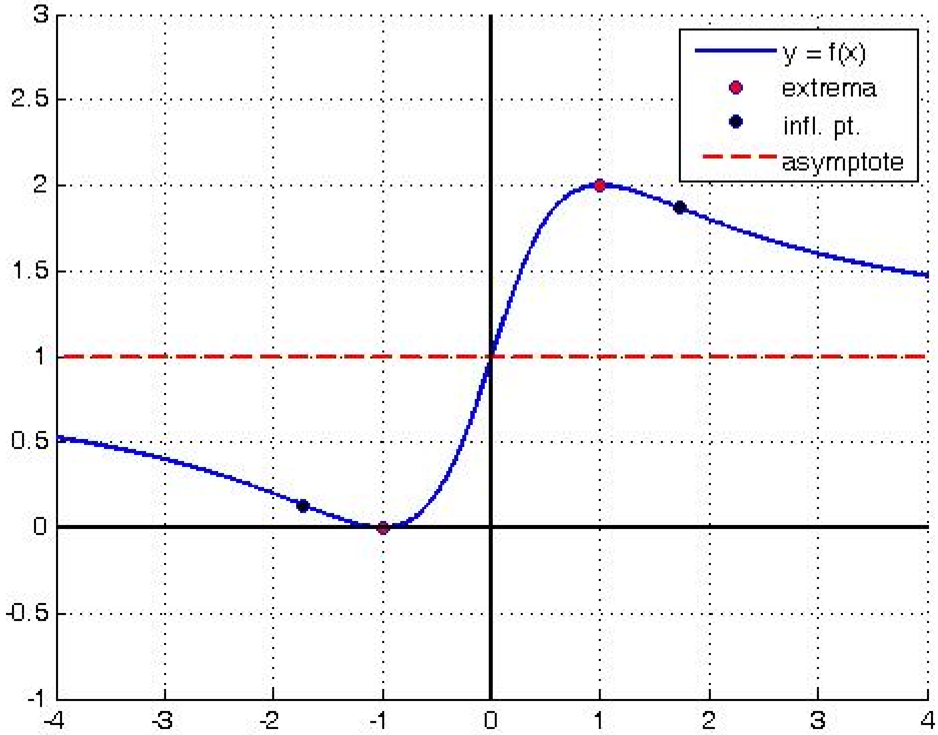
From the chart in (b), we know that

f is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ (1%), concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ (1%).

(d) Since $f(x)$ and $f'(x)$ are finite for any real x, the graph $y = f(x)$ does not have any vertical asymptotes.

Since $\lim_{x\to\pm\infty} f(x) = 1$ (1%), $y = f(x)$ has a horizontal asymptote $y = 1$ (1%).

Sketch of $f(x)$:



6. (10%) Consider all the rectangles with base on the line $y = -2$ and with two upper vertices on the ellipse $x^2 + y^2/4 = 1$ and symmetric with respect to the y-axis. Find the maximal possible area for such a rectangle.

Solution:

$$\begin{aligned}&f(\theta) = 2\cos\theta(2\sin\theta + 2) \ \ (3pt.) \quad 0 \le \theta \le \frac{\pi}{2} \ (1\ pt.)\\ &f'(\theta) = -4(2\sin\theta - 1)(\sin\theta + 1) \ \ (2\,pt.)\\ &\sin\theta = \frac{1}{2}\\ &C.P.\ at\ \theta = \frac{\pi}{6}\end{aligned}$$

$$\begin{aligned}&f(0) \ \ = 4\\ &f(\frac{\pi}{6}) = 3\sqrt{3}\\ &f(\frac{\pi}{2}) = 0\end{aligned}$$

(2 pt.)

$f(\frac{\pi}{6}) = 3\sqrt{3}$ is the maximum (2 pt.)

7. (10%) Find $f'(2)$ given that $f(x) = \displaystyle\int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}}dt$.

Solution:

$f(x) = \displaystyle\int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}}dt = x\int_{2x}^{x^3-4} \frac{1}{1+\sqrt{t}}dt$

By the fundamental theorem of calculus,

$$\frac{d}{dx}f(x) = \int_{2x}^{x^3-4} \frac{1}{1+\sqrt{t}}dt + x\left[\frac{1}{1+\sqrt{x^3-4}} \cdot (3x^2) - \frac{1}{1+\sqrt{2x}} \cdot (2)\right]$$

(3pts) (2pts) (2pts)

$$f'(2) = \int_4^4 \frac{1}{1+\sqrt{t}dt} + 2\left[\frac{3\dot{2}^2}{1+\sqrt{4}} - \frac{2}{1+\sqrt{4}}\right] = \frac{20}{3}$$

(2pts) (1pts)

8. (10%) Calculate $\displaystyle\int \frac{\csc^2 2x}{\sqrt{2+\cot 2x}}dx$.

Solution:

Let $u = 2 + \cot 2x$, then d$u = -2\csc^2 2x$ dx (2 point)

$\displaystyle\int \frac{\csc^2 x}{\sqrt{2+\cot 2x}}$ dx

$= \displaystyle\int \frac{\frac{-1}{2}}{\sqrt{u}}$ du (2 point)

$=\displaystyle\frac{-1}{2}\int \frac{1}{\sqrt{u}}$ du

$=$ - $u^{\frac{1}{2}} + C$ (5points)

$=$ -$\sqrt{\cot 2x + 2} + C$, where C is a constant. (1 point)