1002微甲08-12班期末考解答和評分標準

1. (12%) Evaluate $\int_0^1 \int_0^{1-x} \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\ dzdydx$. [*Hint:* switch $dzdydx$ into $dxdydz$.]

Solution:
We need to compute

$$\int_0^1 \int_0^{1-x} \int_y^1 \frac{\sin(\pi z)}{z(2-z)} dzdydx.$$

The region of integration is bounded by $[0,1]\times[0,1]\times[0,1]\cap\{(x,y,z)\in\mathbb{R}^3 | x+y\leq 1, z\geq y\}$. (4% If you derived the correct region or drawed a correct graph indicating the correct region)

Therefore, $\int_0^1 \int_0^{1-x} \int_y^1 f(x,y,z)dzdydx = \int_0^1 \int_0^{z} \int_0^{1-y} f(x,y,z)dxdydz$. (4%; you won't get full points if you did not provide your reasons)

Hence

$$\begin{aligned}\int_0^1 \int_0^{1-x} \int_y^1 \frac{\sin(\pi z)}{z(2-z)} dzdydx &= \int_0^1 \int_0^{z} \int_0^{1-y} \frac{\sin(\pi z)}{z(2-z)} dxdydz \\ &= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \int_0^z (1-y)dydz \\ &= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} (z-\frac{z^2}{2})dz \\ &= \int_0^1 \frac{1}{2}\sin(\pi z)dz = -\frac{1}{2\pi}\cos(\pi z)\Big|_0^1 = \frac{1}{\pi}. \qquad (4\%)\end{aligned}$$

2. (12%) Evaluate $\iint\limits_R e^{-4x^2+12xy-10y^2}\ dxdy$ where R is the region satisfying $x\geq 2y$ and $y\geq 0$.

[*Hint:* rewrite $4x^2-12xy+10y^2$ as u^2+v^2 by suitable changing of coordinates.]

Solution:

$$Let \quad 4x^2-12xy+10y^2 = u^2+v^2 \text{ (2 points)}$$

Then let $u=2x-3y$ and $v=y$ then

$$\begin{cases} x = \dfrac{u+3v}{2} \\ y = v \end{cases} \quad \text{(2 points)}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2} \text{ (2 points)}$$

$$\begin{aligned}&\iint_R e^{-4x^2+12xy-10y^2}dxdy \\ &= \int_0^\infty \int_0^u e^{-(u^2+v^2)}dvdu \\ &= \int_0^{\frac{\pi}{4}} \int_0^\infty e^{-r^2}\frac{1}{2}rdrd\theta \\ &= \frac{\pi}{16} \text{ (6 points)}\end{aligned}$$

3. (12%) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)^{\frac{1}{2}}\ dzdydx$.

Solution:

The domain is defined by

$x^2 + y^2 + z^2 < 2$, $z^2 > x^2 + y^2$, $0 < y < \sqrt{1-x^2}$, $z > 0$, $x > 0$. (4%)

Change the variable by

$$x = r\sin\phi\cos\theta$$

$$y = r\sin\phi\sin\theta$$

$$z = r\cos\phi$$

And the domain is defined by $0 < r < 2$, $0 < \phi < \dfrac{\pi}{4}$, $0 < \theta < \dfrac{\pi}{2}$. (4%)

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3 \sin\phi \, dr \, d\theta \, d\phi \qquad (2\%)$$

$$= \frac{\pi}{2} \cdot (1 - \frac{1}{\sqrt{2}}) \cdot \frac{\sqrt{2}^4}{4} = \frac{\pi}{2} \cdot (1 - \frac{1}{\sqrt{2}}) \qquad (2\%)$$

4. (12%) Find the work done by the force field

$$\mathbf{F} = (2xyz + z^2 - 2y^2 + 1)\ \mathbf{i} + (x^2z - 4xy + x)\ \mathbf{j} + (x^2y + 2xz + 2z)\ \mathbf{k}$$

in moving a particle along the curve C parameterized by $\mathbf{r}(t) = -2t\ \mathbf{i} + \sin^{-1} t\ \mathbf{j} + \pi t\ \mathbf{k}$, $0 \le t \le 1$.

Solution:

Let $\phi(x,\,y,\,z) = x^2yz + xz^2 - 2xy^2 + x + z^2$, then

$$\nabla\phi = (2xyz + z^2 - 2y^2 + 1)\,\mathbf{i} + (x^2z - 4xy)\,\mathbf{j} + (x^2y + 2x + 2z)\,\mathbf{k}$$

$$\mathbf{F} = \nabla\phi + x\,\mathbf{j} \qquad (4\%)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\nabla\phi + x\,\mathbf{j}) \cdot d\mathbf{r} = \phi(x,\,y,\,z)\Big|_{\mathbf{r}(0)}^{\mathbf{r}(1)} + \int_C x\,dy \qquad (4\%)$$

$$\phi(x,\,y,\,z)\Big|_{\mathbf{r}(0)}^{\mathbf{r}(1)} = \phi(x,\,y,\,z)\Big|_{(0,0,0)}^{(-2,\,\frac{\pi}{2},\,\pi)} = 2\pi^2 - 2\pi^2 + \pi^2 - 2 + \pi^2 = 2\pi^2 - 2$$

$$\int_C x\,dy = \int_0^1 \frac{-2t}{\sqrt{1-t^2}} = 2\sqrt{1-t^2}\Big|_0^1 = -2$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi^2 - 4 \qquad (4\%)$$

5. (12%) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ counterclockwise around the region bounded by $0 \le x \le 1$ and $x^2 \le y \le 1$, where $\mathbf{F}(x,y) = \sin(x^3)\ \mathbf{i} + \dfrac{x^2}{1+y^2}\ \mathbf{j}$.

Solution:

By Green Theorem and Fubini Theorem

$$
\begin{aligned}
&\oint_C \mathbf{F} \cdot d\mathbf{r} \\
=&\iint_R \frac{\partial}{\partial x} \frac{x^2}{1+y^2} - \frac{\partial}{\partial y} \sin(x^3) dA \\
=&\int_0^1 \int_{x^2}^1 \frac{2x}{1+y^2} dydx (6\%) \\
=&\int_0^1 \int_0^{\sqrt{y}} \frac{2x}{1+y^2} dxdy (3\%) \\
=&\int_0^1 \left. \frac{x^2}{1+y^2} \right|_{x=0}^{\sqrt{y}} dy \\
=&\int_0^1 \frac{y}{1+y^2} dy \\
=&\frac{1}{2} \ln(1+y^2) \Big|_0^1 = \frac{1}{2} \ln 2. (3\%)
\end{aligned}
$$

6. (12%) Evaluate the integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{N}} dS$, where

$$\mathbf{F} = (x+y)\ \mathbf{i} + (y + e^{z^2})\ \mathbf{j} + (2x \sin y + 2z)\ \mathbf{k},$$

S is the surface of the region D inside the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$, $\hat{\mathbf{N}}$ is the unit outward normal of S.

Solution:

By Divergence Theorem

$$
\begin{aligned}
&\iint_S \mathbf{F} \cdot \hat{\mathbf{N}} dS \\
=&\iiint_D \text{div}\mathbf{F} dV \\
=&\iiint_D 4dV = 4V (4\%) \\
=&4 \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dz dr d\theta (6\%) \\
=&\frac{16\pi}{3} (8 - 3\sqrt{3}). (2\%)
\end{aligned}
$$

7. (12%) Find the flux of $\mathbf{F} = yz\ \mathbf{i} - xz\ \mathbf{j} + (x^2 + y^2)\ \mathbf{k}$ upward through the surface

$$\mathbf{r}(u, v) = e^u \cos v\ \mathbf{i} + e^u \sin v\ \mathbf{j} + u\ \mathbf{k},$$

where $0 \le u \le 1$, $0 \le v \le \pi$.

Solution:

$$\frac{\partial r}{\partial u} = (e^u cosv, e^u sinv, 1)$$

$$\frac{\partial r}{\partial v} = (-e^u sinv, e^u cosv, 0)$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (-e^u cosv, -e^u sinv, e^{2u}) \qquad (2 \quad points)$$

$$\text{flux} = \int_0^{\pi}\int_0^1 (ue^u sinv, -ue^u cosv, e^{2u})\cdot(-e^u cosv, -e^u sinv, e^{2u})dudv$$

$$= \int_0^{\pi}\int_0^1 e^{4u}dudv$$

$$= \frac{\pi}{4}(e^4-1) \qquad (10 \quad points)$$

8. (12%) Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the curve

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + (\sin t + \cos t)\ \mathbf{k}, \quad 0 \le t \le 2\pi,$$

where $\mathbf{F} = (e^x - y^3)\ \mathbf{i} + (e^y + x^3)\ \mathbf{j} + (e^z + x + y)\ \mathbf{k}$.

Solution:

Method 1: By Stokes' theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{N}ds. \qquad (1\text{ pt})$$

We also have $C : \begin{cases} x^2+y^2=1, \\ z = x+y. \end{cases}$

Let $R : \begin{cases} -x-y+z=0, \\ x^2+y^2 \le 1. \end{cases}$ Hence C is the boundary of R. Its parametric equation is $R(x,y) = x\mathbf{i}+y\mathbf{j}+(x+y)\mathbf{k}$, $x^2+y^2 \le 1$. $\mathbf{N}$ is the upper normal of R.

$$\mathbf{N}ds = \frac{\partial R}{\partial x}\times\frac{\partial R}{\partial y}\,dx\,dy = (-\mathbf{i}-\mathbf{j}+\mathbf{k})dx\,dy \qquad (3\text{ pts})$$

$$\nabla\times\mathbf{F} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x - y^3 & e^y + x^3 & e^z + x + y \end{bmatrix} = \mathbf{i}-\mathbf{j}+(3x^2+3y^2)\mathbf{k} \qquad (3\text{ pts})$$

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \nabla\times\mathbf{F}\cdot\mathbf{N}ds \\ &= \iint_{x^2+y^2\le 1} \big(\mathbf{i}-\mathbf{j}+(3x^2+3y^2)\mathbf{k}\big)\cdot(-\mathbf{i}-\mathbf{j}+\mathbf{k})dx\,dy \\ &= \iint_{x^2+y^2\le 1}(3x^2+3y^2)dx\,dy \\ &= \int_0^{2\pi}\int_{r=0}^1 3r^2\,r\,dr\,d\theta \\ &= 2\pi\cdot 3\cdot \left.\frac{r^4}{4}\right|_0^1 \\ &= \frac{3\pi}{2}. \qquad (5\text{ pts}) \end{aligned}$$

Method 2:

$$d\mathbf{r}(t) = \big(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + (\cos t - \sin t)\mathbf{k}\big)dt \qquad (1\text{ pts})$$

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r}(t) &= \Big((-e^{\cos t} + \sin^3 t)\sin t + (e^{\sin t} + \cos^3 t)\cos t \\ &\quad +(e^{\sin t + \cos t} + \cos t + \sin t)(\cos t - \sin t)\Big)dt \qquad (2\text{ pts}) \\ &= e^{\cos t}d\cos t + \sin^4 t\,dt + e^{\sin t}d\sin t + \cos^4 t\,dt \\ &\quad +e^{\sin t+\cos t}d(\sin t + \cos t) + (\cos^2 t - \sin^2 t)dt \qquad (2\text{ pts}) \end{aligned}$$

$$
\begin{aligned}
\int_{t=0}^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r}(t) &= \left(e^{\cos t}+e^{\sin t}+e^{\sin t+\cos t}\right)_{t=0}^{2\pi} + \int_{t=0}^{2\pi} \sin^4 t+\cos^4 t+\cos^2 t-\sin^2 t\, dt \\
&= 0 + 2\int_{t=0}^{2\pi} \sin^4 t\, dt + 0, \quad \text{since } \int_{t=0}^{2\pi} \sin^n t\, dt = \int_{t=0}^{2\pi} \cos^n t\, dt \\
&= 2\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right)^2 dt \\
&= \int_{t=0}^{2\pi} \frac{1}{2} - \cos 2t + \frac{1}{2}\cos^2 2t\, dt \\
&= \pi + \frac{1}{8}\int_{t=0}^{2\pi} \frac{1+\cos 4t}{2} d4t, \quad \text{since } \int_0^{2\pi n} \cos s\, ds = 0 \\
&= \pi + \frac{\pi}{2} \\
&= \frac{3\pi}{2} \qquad \text{(7 pts)}
\end{aligned}
$$

9. (12%) Find the general solution of the differential equation

$$y'' + 3y' + 2y = 2e^{-x} + 4e^{-x}\cos x.$$

Solution:

First, we consider the homogeneus equation $y''+3y'+2y=0$, try $y=e^{rx}$, we obtain the auxiliary equation $r^2+3r+2=0$, so $(r+1)(r+2)=0$, thus we get $r=-1,-2$.
So the general solution to the homogeneous equation is $y_h = C_1e^{-x}+C_2e^{-2x}$, C_1, C_2 are constants
Since the differential equation is nonhomogeneus , the solution to this equation is given by $y = y_h + y_p$, where y_p is the particular solution to this equation .
To compute y_p.

Method I
we need to solve the following two equations
$y''+3y'+2y = 2e^{-x}$ (1)
$y''+3y'+2y = 4e^{-x}\cos x$ (2)
For (1), since $y=e^{-x}$ is already a solution to this differertial equation, we try $y = Axe^{-x}$, and we get $A = 2$, so the particular solution to (1) is given by $y_{p1} = 2xe^{-x}$
For (2), we try $y = Be^{-x}\cos x + Ce^{-x}\sin x$, and we get $B=-2, C=2$, so the particular solution to (2) is given by $y_{p2} = -2e^{-x}\cos x + 2e^{-x}\sin x$.
Since this is a linear differential equation, we have the particular solution to $y''+3y'+2y = 2e^{-x}+4e^{-x}\cos x$, say y_p, is the sum of the particular solution of (1) and (2), i.e., $y_p = y_{p1}+y_{p2} = 2xe^{-x} - 2e^{-x}\cos x + 2e^{-x}\sin x$
So, the solution to this equation is given by $y = y_h + y_p = C_1e^{-x} + C_2e^{-2x} + 2xe^{-x} - 2e^{-x}\cos x + 2e^{-x}\sin x$
You can also try $y_p = Axe^{-x} + Be^{-x}\cos x + Ce^{-x}\sin x$ directly to solve A, B, C

Method II
The second method use variation of paramrters. For the homogeneus equation $y''+3y'+2y=0$, from the above calculation, we have there are two linearly independent solutions , say $y_1 = e^{-x}$, $y_2 = e^{-2x}$, and we search the particular solution of the form $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$.
For simplicity, we choose y_p to satisfy the differential equation $u_1'y_1 + u_2'y_2 = 0$, and by this assumption , for this y_p need to satisfy this differential equation $y''+3y'+2y = 2e^{-x}+4e^{-x}\cos x$, we must have $u_1'y_1' + u_2'y_2' = 2e^{-x}+4e^{-x}\cos x$, so we have following two equations,

$u_1'y_1 + u_2'y_2 = u_1'e^{-x} + u_2'e^{-2x} = 0$ (3)
$u_1'y_1' + u_2'y_2' = -u_1'e^{-x} - 2u_2'e^{-2x} = 2e^{-x}+4e^{-x}\cos x$ (4)

$(3)*2+(4) \to u_1'e^{-x} = 2e^{-x}+4e^{-x}\cos x \to u_1' = 2+4\cos x$
,so , $u_1 = 2x + 4\sin x + C_3$ for som constant C_3.
$(3)+(4) \to -u_2'e^{-2x} = 2e^{-x}+4e^{-x}\cos x \to u_2' = -2e^{x} - 4e^{x}\cos x$.

Using integration by parts, we get $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$, so $\int e^x \cos x dx = \frac{1}{2}(e^x \sin x + e^x \cos x)$. Thus, $u_2 = -2e^x - 2(e^x \sin x + e^x \cos x) + C_4$, where C_4 is a constant.
So, $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1(x)e^{-x} + u_2(x)e^{-2x}$
$= (2x + 4\sin x + C_3)e^{-x} + (-2e^x - 2(e^x \sin x + e^x \cos x) + C_4)e^{-2x}$
$= 2xe^{-x} - 2e^{-x} + 2e^{-x}\sin x - 2e^{-x}\cos x + C_3e^{-x} + C_4e^{-2x}$ Thus the general solution is
$y = y_h + y_p == C_1e^{-x} + C_2e^{-2x} + 2xe^{-x} - 2e^{-x} + 2e^{-x}\sin x - 2e^{-x}\cos x + C_3e^{-x} + C_4e^{-2x}$
$= C_5e^{-x} + C_6e^{-2x} + 2xe^{-x} + 2e^{-x}\sin x - 2e^{-x}\cos x$. for some constant C_5, C_6.

10. (12%) Solve the differential equation

$$x^3y''' + xy' - y = 0$$

in the interval $x > 0$.

Solution:
Let $y = x^r$(2%) ,then we substitute it.

We have $x^r[(r-1)r(r-2) + r - 1] = 0$.

$\Longrightarrow [r-1]^3 = 0$

$\Longrightarrow r = 1, 1, 1$(7%)

Then by **"checking detail"**(2%) the solution is

$y = c_1x + c_2x\ln x + c_3x(\ln x)^2$(1%).