1002微甲01-05班期中考解答和評分標準

1. (16 points) Prove that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \frac{1}{2^n}$ converges absolutely, and find its sum.

Solution:

Part I (6pts) Prove the series converges absolutely

let $a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \frac{1}{2^n}$

then the series convergent absolutely if $\sum_{n=2}^{\infty} |a_n|$ converges

use ratio test:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = (\frac{1}{(n+1)n2^{n+1}} / \frac{1}{(n)(n)2^n}) = \frac{1}{2} < 1$$

use comparison test:

compared a_n to $b_n = \frac{1}{n^2}$ or $b_n = \frac{1}{2^n}$

show that $a_n < b_n$ everywhere and prove that $\sum_{n=2}^{\infty} |b_n|$ converges

Part II (10pts)find its sum

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 .... = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$g(x) = \int \frac{1}{1+x} dx = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 .... + c1 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n + c2$$

For $x = 0 \to c1 = c2 = 0$

$$h(x) = \int \ln(1+x) dx = \frac{1}{1\times 2}x^2 - \frac{1}{2\times 3}x^3 + \frac{1}{3\times 4}x^4 + ... + c3 = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n + c4$$

$$(1+x)\ln(1+x) - (1+x) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n + c4$$

For $x = 0 \to c4 = -1$

let $x = \frac{1}{2}$

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n = \frac{3}{2} \ln \frac{3}{2} - \frac{1}{2}$$

2. (a) (8 points) The Binomial Theorem implies that

$$(1-x)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{k^n (n!)^2} x^n$$

for some constant k. Find k, and find the interval of convergence of the power series.

(b) (8 points) Estimate the error if one uses $x = -\frac{1}{4}$, and the first five non-zero terms in (a) to approximate $\frac{1}{\sqrt{5}}$.

Solution:

(a) $(1-x)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n n!^2} x^n$. So $k = 4$.(2 points)
Let $a_n = \frac{(2n)!}{4^n n!^2}$. $R = \lim \frac{a_n}{a_{n+1}} = 1$. (2 points)
At $x = -1$, since a_n is decreasing and converge to 0, $\sum a_n (-1)^n$ converges. (2 points)
At $x = 1$, since $\sum a_n \geq \sum a_n x^n = (1-x)^{\frac{-1}{2}}$ for all $x \in (-1, 1)$ and $\lim_{x \to 1^-} (1-x)^{\frac{-1}{2}} = \infty$.
$\sum a_n$ diverges. (2 points)
Hence, the interval of convergence is $[-1, 1)$.
(b) $\frac{1}{\sqrt{5}} = \frac{1}{2}((1-(-\frac{1}{4}))^{\frac{-1}{2}} = \frac{1}{2}(\sum_{n=0}^{\infty} \frac{(2n)!}{4^n n!^2}(-\frac{1}{4})^n)$ (1 point)
Since it is an alternating series (2 points),
$|\frac{1}{\sqrt{5}} - \frac{1}{2}(\sum_{n=0}^{\infty} \frac{(2n)!}{4^n n!^2}(-\frac{1}{4})^n)| \leq \frac{10!}{2 * 4^{10}(5!)^2} = \frac{126}{4^{10}}$(5 points)

3. (16 points) Consider the curve

$$\mathbf{r}(t) = t^2\ \mathbf{i} + (\sin t - t\cos t)\ \mathbf{j} + (\cos t + t \sin t)\ \mathbf{k},\ \ t \geq 0$$

Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, the curvature $\kappa(t)$ and the torsion $\tau(t)$.

Solution:
$r(t) = (t^2, sint - tcost, cost + tsint)$

$r'(t) = v(t) = (2t, cost + tsint - cost, -sint + tcost + sint) = (2t, tsint, tcost)$ (1%)
$r''(t) = a(t) = (2, sint + tcost, cost - tsint)$
$r'''(t) = a'(t) = (0, 2cost - tsint, -2sint - tcost)$

$|r'(t)| = \sqrt{4t^2 + t^2 sin^2 t + t^2 cos^2 t} = \sqrt{5}t$ (1%)
$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{5}}(2, sint, cost) = \frac{\sqrt{5}}{5}(2, sint, cost)$ (2%)
To get full points (3%) please answer both question correctly.

$T'(t) = \frac{1}{\sqrt{5}}(0, cost, -sint)$, $|T'(t)| = \frac{1}{\sqrt{5}}\sqrt{cos^2 t + sin^2 t} = \frac{1}{\sqrt{5}}$
$N(t) = \frac{T'(t)}{|T'(t)|} = B(t) \cdot N(t) = (0, cost, -sint)$ (3%)
To get full points (3%) please answer both question correctly.

$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{|v(t) \times a(t)|}{[v(t)]^3} = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{5}t} = \frac{1}{5t}$ (3%)
To get full points (3%) please answer both question correctly.

$v(t) \times a(t) = (-t^2, 2t^2 sint, 2t^2 cost)$ (1%)
$|v(t) \times a(t)| = \sqrt{(-t^2)^2 + (2t^2 sint)^2 + (2t^2 cost)^2} = \sqrt{t^4 + 4t^4} = \sqrt{5}t^2$
If your calculation is wrong, you will receive some partial credit.
$B(t) = \frac{v(t) \times a(t)}{|v(t) \times a(t)|} = \frac{1}{\sqrt{5}}(-1, 2sint, 2cost) = \frac{\sqrt{5}}{5}(-1, 2sint, 2cost)$ (2%)
To get full points (3%) please answer both question correctly.

$\tau(t) = \frac{[v(t) \times a(t)] \cdot a'(t)}{|v(t) \times a(t)|^2} = \frac{2t^2 sint(2cost - tsint) + 2t^2 cost(-2sint - tcost)}{5t^4}$
$= \frac{(4t^2 sint \cdot cost - 2t^3 sin^2 t) + (-4t^2 sint \cdot cost - 2t^2 cost^2 t)}{5t^4}$
$= \frac{-2t^3}{5t^4} = \frac{-2}{5t}$ (3%)

I will also grant partial credit for partial solutions and solutions with minor flaws. I will give no credit for wildly incorrect answers which are obviously only there in the hopes of getting partial credit.

4. (16 points) Let $F(x,y,z) = x^2 + 2z + \int_y^z \sqrt[3]{(t^2+7)y^2}\,dt$. Find the tangent plane of the surface $F(x,y,z) = 2$ at the point $(2,-1,-1)$.

Solution:

$F1 = \dfrac{\partial F}{\partial x} = 2x$ (1 points)

$F2 = \dfrac{\partial F}{\partial y} = \dfrac{2}{3} y^{\frac{-1}{3}} \displaystyle\int_y^z \sqrt[3]{t^2+7}\,dt - y^{\frac{2}{3}} \sqrt[3]{y^2+7}$ (5 points)

$F3 = \dfrac{\partial F}{\partial z} = 2 + y^{\frac{2}{3}} \sqrt[3]{z^2+7}$ (4 points)

$F1(2,-1,-1) = 4$ (1 points)

$F2(2,-1,-1) = -2$ (2 points)

$F3(2,-1,-1) = 4$ (1 points)

$4(x-2) - 2(y+1) + 4(z+1) = 0$ (2 points)

5. Suppose that $z = f(x,y)$ has continuous second order partial derivatives, and $x = s^2 - t^2$, $y = 2st$. Define

$$F(s,t) = f(s^2 - t^2, 2st)$$

(a) (6 points) Express F_s, F_t in terms of f_x, f_y, s and t.

(b) (10 points) Show that $F_{ss} + F_{tt} = h(s,t)(f_{xx} + f_{yy})$ for some function $h(s,t)$. Find $h(s,t)$ explicitly.

Solution:

5.(a)

$$F_s(s,t) = 2sf_x(s^2-t^2,2st) + 2tf_y(s^2-t^2,2st) \ldots\ldots (3pts)$$
$$F_t(s,t) = -2tf_x(s^2-t^2,2st) + 2sf_y(s^2-t^2,2st) \ldots\ldots (3pts)$$

(b)

$$\begin{aligned} F_{ss}(s,t) &= 2f_x + 2s[2sf_{xx} + 2tf_{xy}] + 2t[2sf_{xy} + 2tf_{yy}] \\ &= 2f_x + 4s^2 f_{xx} + 8stf_{xy} + 4t^2 f_{yy} \ldots\ldots (3pts) \\ F_{tt}(s,t) &= -2f_x + 4t^2 f_{xx} - 8stf_{xy} + 4s^2 f_{yy} \ldots\ldots (3pts) \end{aligned}$$

$$\begin{aligned} F_{ss}(s,t) + F_{tt}(s,t) &= 4(s^2+t^2)f_{xx} + 4(s^2+t^2)f_{yy} \ldots\ldots (2pts) \\ &= 4(s^2+t^2)(f_{xx} + f_{yy}) \Rightarrow h(s,t) = 4(s^2+t^2) \ldots\ldots (2pts) \end{aligned}$$

6. Let $f(x,y,z) = yx^2 + xz^2 - y$.

(a) (10 points) Find all critical points of $f(x,y,z)$ and classify them.

(b) (10 points) Find the maximum and minimum of f on the region $x^2 + y^2 + z^2 \leq 1$.

Solution:

(a)

$f_1 = 2xy + z^2$ (1 point)
$f_2 = x^2 - 1$ (1 point)
$f_3 = 2xz$ (1 point)

$\Rightarrow$ critical points at $f_1 = f_2 = f_3 = 0$
$\Rightarrow (\pm 1, 0, 0)$ (2 points)

$$\text{Hessian} \begin{pmatrix} 2y & 2x & 2z \\ 2x & 0 & 0 \\ 2z & 0 & 2x \end{pmatrix} \text{(1 point)} \Rightarrow \begin{pmatrix} 0 & \pm 2 & 0 \\ \pm 2 & 0 & 0 \\ 0 & 0 & \pm 2 \end{pmatrix} \text{(2 points)}$$

def H $\neq 0 \Rightarrow$ neither positive definite nor negative definite
$\Rightarrow$ both saddle points (2 points)

(b)

$$x^2+y^2+z^2-1=g(x,y,z)\le 0$$

$$\text{Solve the system}\begin{cases}\nabla f=\lambda\nabla g \text{ (1 point)}\\ g=0\end{cases}$$

$$\begin{cases}2xz+z^2=2\lambda x & \text{(1 point)}\\ x^2-1=2\lambda y & \text{(1 point)}\\ 2xz=2\lambda z & \text{(1 point)}\\ x^2+y^2+z^2=1\end{cases}$$

case 1: $z=0$, then $2xy=2\lambda x$. If $x=0$, $y=\pm1\Rightarrow(0,\pm1,0)\Rightarrow f(0,\pm1,0)=\mp1$
If $x\neq0$, $y=\lambda\Rightarrow x^2=2y^2+1=1-y^2\Rightarrow(\pm1,0,0)\Rightarrow f(\pm1,0,0)=0$ (1 point)
case 2: $z\neq0$, $x=\lambda\Rightarrow x^2=2xy+1=(2x^2-z^2)+1\Rightarrow(0,0,\pm1)\Rightarrow f(0,0,\pm1)=0$ (1 point)
max $=1$ at $(0,-1,0)$ (2 points), min $=-1$ at $(0,1,0)$ (2 points)