1. (13%) Let $f(x) = \dfrac{\cot \frac{\pi}{2} x}{\ln x}$, $0 < x < 1$. Find $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^-} f'(x)$.

   Sol:

$$f(x) := \frac{\cot(\frac{\pi}{2}x)}{\ln x}$$

$$\lim_{x\to 1^-} f(x) \overset{[\frac{0}{0}],L'H}{=} \lim_{x\to 1^-} \frac{-\frac{\pi}{2}\csc^2(\frac{\pi}{2}x)}{\frac{1}{x}} \overset{x:=1}{=} \frac{-\frac{\pi}{2}}{1} = -\frac{\pi}{2}$$

$$f'(x) = \frac{(\ln x)(-\frac{\pi}{2}\csc^2(\frac{\pi}{2}x)) - \cot(\frac{\pi}{2}x)\frac{1}{x}}{(\ln x)^2}$$

$$\begin{aligned}
\lim_{x\to 1^-} f'(x) &= \lim_{x\to 1^-} \frac{(\ln x)(-\frac{\pi}{2}\csc^2(\frac{\pi}{2}x)) - \cot(\frac{\pi}{2}x)\frac{1}{x}}{(\ln x)^2} \\
&= \lim_{x\to 1^-} \frac{-x(\ln x)\frac{\pi}{2} - \cos(\frac{\pi}{2}x)\sin(\frac{\pi x}{2})}{(\ln x)^2} * \lim_{x\to 1^-} \frac{1}{x\sin^2\frac{\pi x}{2}} \\
&\overset{[\frac{0}{0}],L'H}{=} \lim_{x\to 1^-} \frac{\frac{-\pi}{2}(\frac{x}{x} + \ln x) + \frac{\pi}{2}\sin^2\frac{\pi x}{2} - \frac{\pi}{2}\cos^2\frac{\pi x}{2}}{\frac{2}{x}\ln x} * 1 \\
&= \lim_{x\to 1^-} \frac{\frac{\pi}{2}(-1 - \ln x + \sin^2\frac{\pi x}{2} - \cos^2\frac{\pi x}{2})}{2\ln x} * \lim_{x\to 1^-} x \\
&= \lim_{x\to 1^-} \frac{\frac{\pi}{2}(-\ln x - 2\cos^2\frac{\pi x}{2})}{2\ln x} * 1 \\
&\overset{L'H}{=} \lim_{x\to 1^-} \frac{\pi(-\frac{1}{x} + 4\frac{\pi}{2}\cos\frac{\pi x}{2}\sin\frac{\pi x}{2})}{4\frac{1}{x}} \\
&\overset{\cos\frac{\pi}{2}=0}{=} \frac{-\pi}{4}
\end{aligned}$$

2. (9%) Find the limit $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

   Sol:

Take the natural logarithm, we consider

$$\begin{aligned}
&\lim_{x\to 0}\frac{\ln|\tan x|-\ln|x|}{x^2}\\
=&\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}\\
=&\lim_{x\to 0}\frac{\frac{1}{\sin x\cos x}-\frac{1}{x}}{2x}\\
=&\lim_{x\to 0}\frac{x-\sin x\cos x}{2x^2\sin x\cos x}\\
=&\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{4x\sin x\cos x+2x^2\cos^2 x-2x^2\sin^2 x}\\
=&\lim_{x\to 0}\frac{2\sin^2 x}{4x\sin x\cos x+2x^2\cos^2 x-2x^2\sin^2 x}\\
=&\lim_{x\to 0}\frac{\sin^2 x}{2x\sin x\cos x+x^2\cos^2 x-x^2\sin^2 x}\\
=&\lim_{x\to 0}\frac{1}{2\frac{x}{\sin x}\cos x+\left(\frac{x}{\sin x}\right)^2\cos^2 x-x^2}\\
=&\frac{1}{2+1-0}=\frac{1}{3}
\end{aligned}$$

So the answer is $e^{\frac{1}{3}}$.

3. (9%) Let $f(x)=\begin{cases}\tan^{-1}\left|\dfrac{2x-1}{2x+1}\right|, & x<0,\\ ax+b, & x\geq 0.\end{cases}$ Find $a,\ b$ that make $f(x)$ differentiable at $x=0$.

Sol:

If f is differentiable at $x=0$, it has to be continuous at $x=0$.

$\lim\limits_{x\to 0^-}=\tan^{-1}\left|\dfrac{2x-1}{2x+1}\right|=\dfrac{\pi}{4}=f(0)=b$

$\Longrightarrow b=\dfrac{\pi}{4}$

$$\begin{aligned}
f'_-(0)=&\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}\\
=&\lim_{x\to 0^-}\frac{\tan^{-1}\left|\frac{2x-1}{2x+1}\right|-\frac{\pi}{4}}{x}\left(\frac{0}{0}\right)\\
=&\lim_{x\to 0^-}\frac{\frac{sgn(\frac{2x-1}{2x+1})\cdot(\frac{4}{(2x+1)^2})}{1+(\frac{2x-1}{2x+1})^2}}{1}\\
=&\lim_{x\to 0^-}\frac{(-1)\cdot 4}{(2x+1)^2+(2x-1)^2}=-2=f'_+(0)=a
\end{aligned}$$

$(x\to 0\Longrightarrow sgn(\dfrac{2x-1}{2x+1})=-1)$

$\Longrightarrow a=-2$

4. (9%) Suppose $f$ is a continuous function on $[a,b]$ and $a\leq f(x)\leq b$ for all $x\in[a,b]$. Show that there exists $c\in[a,b]$ such that $f(c)=c$.

Sol:

$f$ is continuous on $[a,b], a \leq f(x) \leq b$

Let $g(x) = f(x) - x$, $g$ is continuous on $[a,b]$

$g(a) = f(a) - a \geq 0$

$g(b) = f(b) - b \leq 0$

$g(a) \cdot g(b) \leq 0$

By Inter-Mediate Theorem(IVT),there exist $c \in (a,b)$,

$s.t, g(c) = 0 = f(c) - c$

That is $f(c) = c$.

5. (13%) (a) Show that the function $f(x) = x^x$ is strictly increasing on $(e^{-1}, \infty)$.

   (b) If $g$ is the inverse function to $f$ of part (a), find $\lim_{y\to\infty} \frac{g(y)\ln\ln(y)}{\ln y}$.

Sol:

(a) Differentiate $f(x) = x^x = e^{x\ln x}$, we get:

   $f'(x) = x^x(\ln x + 1)$.

   When $x > 0$, $x^x > 0$; also, if $x > e^{-1}$, $\ln x + 1 > 0$.

   Therefore, $f'(x) > 0$ if $x > e^{-1}$.

   Since $f'(x)$ is positive on $(e^{-1}, \infty)$, $f$ is strictly increasing.

(b) Let $y = x^x$, then $\ln y = x\ln x \Rightarrow \ln\ln y = \ln x + \ln\ln x$.
   So $\frac{x\ln\ln y}{\ln y} = \frac{x\ln x}{x\ln x} + \frac{x\ln\ln x}{x\ln x} = 1 + \frac{\ln\ln x}{\ln x}$.
   Since $f$ is strictly increasing on $(e^{-1}, \infty)$, $y \to \infty \Leftrightarrow x \to \infty$.
   Thus $\lim_{y\to\infty} \frac{g(y)\ln\ln y}{\ln y} = 1 + \lim_{x\to\infty} \frac{\ln\ln x}{\ln x} = 1$.

6. (9%) Suppose that the function $y = f(x)$ is defined by the equation $x^4 = x^2 - (y-x)^2,\ x > 0$.

Find the critical point of $y = f(x)$.

Sol:

Differentiate each side with respect to $x$.

$$4x^3 = 2x - 2(y-x)(y'-1)$$

$$y' = \frac{2x^3 - y}{x - y}$$

First, we know that $y' = 0$ if $y = 2x^3$, $x \neq y$.

And Substitute $y$ as $2x^3$ in the equation $x^4 = x^2 - (y-x)^2$. We get

$$4x^6 - 3x^4 = 0$$

$$x = 0,\ \pm\frac{\sqrt{3}}{2}$$

But $x > 0$ so we know that $y' = 0$ if $x = \dfrac{\sqrt{3}}{2}$

$$y = 2x^3 = \frac{3\sqrt{3}}{4}$$

And second, we know that $y'$ might not exist when $x = y$.

Substituting $y$ as $x$ in the equation $x^4 = x^2 - (y-x)^2$. We get

$$x^4 = x^2$$

Provided that $x > 0$. We can observe that $y'$ dose not exist when

$$x = y = 1$$

So the critical points are $(\dfrac{\sqrt{3}}{2}, \dfrac{3\sqrt{3}}{4})$, $(1, 1)$

7. (9%) The point $P$ moves so that at time $t$, $t > 0$, it is at the intersection of the curves $xy + 2x = 2t$ and $y = x^2 t$. How fast is the distance of $P$ from the origin changing at time $t = 2$?

Sol:

$xy + 2x = 2t$ and $y = x^2 t$ for $t = 2$ imply $2x^3 + 2x - 4 = 2(x-1)(x^2+x+2) = 0$,that is $x = 1$ and $y = 2$. Then differentiate $xy + 2x = 2t$,$y = x^2 t$(View $(x, y)$ as a function of time $t$) imply

$$x'y + xy' + 2x' = 2$$

$$y' = 2xx't + x^2.$$

By substitute $(x, y) = (1, 2)$ imply

$$4x' + y' = 2, y' - 4x' = 4,$$

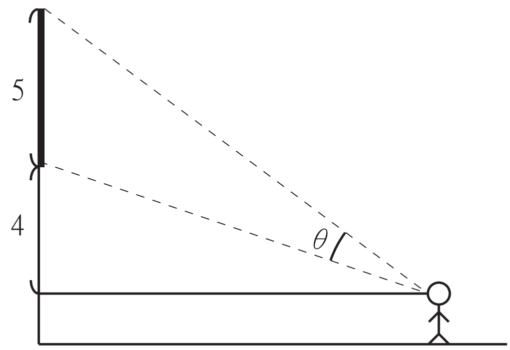
i.e, $x'(2) = 1/8, y'(2) = 3/2$ for $t = 2$. Define the distance between P to the origin as $D = \sqrt{x^2 + y^2}$. The change rate is

$$D' = \frac{xx' + yy'}{\sqrt{x^2 + y^2}}.$$

Substitute above information then we get

$$D' = \frac{1 \cdot 1/8 + 2 \cdot 3/2}{\sqrt{1^2 + 2^2}} = \frac{5\sqrt{5}}{8}$$

8. (9%) How far back from a picture on the wall should one stand to view it best if the picture is 5 m height and the bottom of it is 4 m above eye level?




Sol:

$\theta = \tan^{-1} \frac{9}{x} - \tan^{-1} \frac{4}{x}$

$\frac{d\theta}{dx} = \frac{\frac{-9}{x^2}}{1 + \frac{81}{x^2}} - \frac{\frac{-4}{x^2}}{1 + \frac{16}{x^2}} = \frac{4}{x^2 + 16} - \frac{9}{x^2 + 81} = \frac{-5x^2 + 180}{(x^2 + 16)(x^2 + 81)}$

$x = 6 \Leftrightarrow \frac{d\theta}{dx} = 0$

if $0 < x < 6$ then $\frac{d\theta}{dx} > 0$

if $6 < x$ then $\frac{d\theta}{dx} < 0$

$\Rightarrow \theta$ has a maximum at $x = 6$

9. (20%) Let $f(x) = (x - 1)^{\frac{2}{3}} - (x + 1)^{\frac{2}{3}}$.

   (a) Find all asymptotes.

   (b) Find the intervals of increasing and decreasing.

   (c) Find the intervals of concavity.

   (d) Find all local extreme values and all inflection points.

   (e) Sketch the graph.

(f) Determine the range of real number $c$ for the equation $(x-1)^{\frac{2}{3}} - (x+1)^{\frac{2}{3}} = c$ has zero, one, or two solutions, respectively.

Sol:

(a) consider

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} (x-1)^{2/3} - (x+1)^{2/3} \\ &= \lim_{x\to\pm\infty} \frac{-2x}{(x-1)^{4/3} + (x-1)^{2/3}(x+1)^{1/3} + (x+1)^{4/3}} = 0\end{aligned}$$

(since the rate of $x^{4/3} \gg x$)

i.e. f(x) has the horizontal asymptotes $y = 0$.

(b) since $f'(x) = \frac{2}{3}\ [(x-1)^{-1/3} - \ (x+1)^{-1/3}]$,

we have $f'(x) > 0$ for $|x| > 1$.

for $f'(x) < 0$ for $|x| < 1$.

Then the facts: **increase** on $(-\infty, -1) \cup (1, \infty)$.

**decrease** on $(-1, 1)$.

(c) since $f''(x) = \frac{2}{9}[(x+1)^{-4/3} - \ (x-1)^{-4/3}]$

for $f''(x) = 0 \Rightarrow x = 0$ .

$f''(x) > 0 \Rightarrow (-\infty, 0) - \{-1\}$ .

$f''(x) < 0 \Rightarrow (0, \infty) - \{1\}$ .

(because the singular points of f(x) are at $x = \pm 1$)

Then we have that f(x) is concave up on $(-\infty, -1) \cup (-1, 0)$

concave down on $(0, 1) \cup (1, \infty)$.

(d) By part (b) we know that f(x) has local **maximum** at $x = -1$ and local **minimum** at $x = 1$ .

$\Rightarrow$ Max $= f(-1) = (4)^{1/3}$ min$= f(1) = -(4)^{1/3}$.

On the other hands ,by (c) we have the inflection point is $(0, 0)$.

(e) By the above we can sketch the graph as follow

(f) Consider the graph as above with the intersection of horizontal lines,we have the facts:

*No roots* as $\mid c \mid > (4)^{1/3}$.

*One root* as $c = 0$ *or* $(4)^{1/3}$ *or* $-(4)^{1/3}$.

*Two roots* as $\mid c \mid < (4)^{1/3}$ , $c \neq 0$.