1001微甲01-05班期中考

1. (14%, 7% each) Evaluate the following limits:

   (a) $\lim_{x\to 0}(\frac{\sin^{-1} x}{x})^{\frac{1}{x^2}}$.

   (b) $\lim_{x\to 0^+}(x^{x^x} - 1)$.

   Sol:

   (a) let $x = \sin y$

   as $x \to 0$ means $y \to 0$

   as $y \to 0$

$$e^{\frac{1}{\sin^2 y}\log\frac{y}{\sin y}} = e^{\frac{1}{\sin^2 y}(\log\frac{y}{\sin y} - \log 1)}$$

   by mean value Th , $1 < c < \dfrac{y}{\sin y} \to 1$

$$= e^{\frac{1}{\sin^2 y}(1/c)(\frac{y}{\sin y} - 1)} = e^{(1/c)\frac{y - \sin y}{\sin^3 y}} = e^{(1/c)\frac{y^3}{\sin^3 y}\frac{y - \sin y}{y^3}}$$

   because $1/c \to 1$ and $\dfrac{y^3}{\sin^3 y} \to 1$ and $\dfrac{y - \sin y}{y^3} \to 1/6$ by taylor expansion

$$= e^{1/6}$$

   (b)

$$x^{x^x} = e^{e^{x\log x}\log x} - 1$$

   $x\log x \to 0$ and $logx \to -\infty$

$$e^{e^{x\log x}\log x} - 1 = e^{-\infty} - 1 = -1$$

   $x^x \to 1$

   $x^{x^x} = x^1 = 0$

2. (12%, 6% each)

   (a) Find $\dfrac{d^3}{dx^3}\Big(\dfrac{x}{\sqrt[3]{1+x}}\Big)$.

   (b) Find $\dfrac{d}{dx}\Big(\ln(\cos^{-1}\dfrac{1}{\sqrt{x}})\Big)$.

Sol:

(a) by $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$

$$3x'(1+x)^{-1/3''} + x(1+x)^{-1/3'''} = 3(-1/3)(-4/3)(1+x)^{-7/3} + x(-1/3)(-4/3)(-7/3)(1+x)^{10/3}$$
$$= (4/3)(1+x)^{-7/3} - (28/27)x(1+x)^{10/3}$$

the original order is $1 - 1/3 = 2/3$, after three differentiation, it should be $2/3 - 3 = -7/3$

(b) by chain rule

$$\frac{1}{cos^{-1}1/\sqrt{x}} \frac{-1}{\sqrt{1-1/x}} \frac{-x^{-3/2}}{2}$$

3. (16%) Let $H(x) = \begin{cases} \frac{1}{\pi}\tan^{-1}(ax + \frac{b}{x}), & \text{if } x > 0, \\ c, & \text{if } x = 0, \\ (1 - \ln 2^x)^{\frac{1}{x}}, & \text{if } x < 0. \end{cases}$

(a) Find conditions of a, b, and c such that H is continuous. (6%)

(b) Find conditions of a, b, and c such that H is differentiable. (10%)

Sol:

(a)

$$\lim_{x \to 0^-} H(x) = \lim_{x \to 0^-} (1 - \ln 2^x)^{\frac{1}{x}}$$
$$= \lim_{x \to 0^-} e^{\frac{\ln(1 - x\ln 2)}{x}}$$
$$= e^{\lim_{x \to 0^-} \frac{\ln(1 - x\ln 2)}{x}}$$
$$= e^{-\ln 2} = \frac{1}{2}$$

$$\lim_{x \to 0^+} H(x) = \lim_{x \to 0^+} \frac{1}{\pi}\tan^{-1}(ax + \frac{b}{x})$$
$$= \frac{1}{2} \quad \text{if } b > 0$$
$$= 0 \quad \text{if } b = 0$$
$$= \frac{-1}{2}0 \quad \text{if } b < 0$$

$H(x)$ continuous at 0 $\iff$, $a \in R$,$b > 0$, $c = \frac{1}{2}$

(b) $\lim_{x\to 0-} \dfrac{H(x)-H(0)}{x-0} = \lim_{x\to 0-} \dfrac{(1-\ln 2^x)^{\frac{1}{x}} - \frac{1}{2}}{x-0} = -\dfrac{(\ln 2)^2}{4}$

$\lim_{x\to 0+} \dfrac{H(x)-H(0)}{x-0} = \lim_{x\to 0+} \dfrac{\frac{1}{\pi}\tan^{-1}(ax+\frac{b}{x}) - \frac{1}{2}}{x} = \dfrac{-1}{\pi b}$

$\Longleftrightarrow b = \dfrac{4}{\pi(\ln 2)^2}$

$H(x)$ differentiable at 0 when $a \in R$, $b = \dfrac{4}{\pi(\ln 2)^2}$, $c = \dfrac{1}{2}$

4. (14%) Find the line normal to the curve defined by $\tan^{-1}(xy) + \ln(x+y) = x^y - 1$ at $(x,y) = (1,0)$. Also find y'' of the curve at $(1,0)$.

Sol:

$$\tan^{-1}(xy) + \ln(x+y) = x^y - 1$$

$$\frac{d\tan^{-1}(xy)}{dx} + \frac{d\ln(x+y)}{dx} = \frac{de^{y\ln x}}{dx}$$

$$\frac{1}{(xy)^2+1}(y+xy') + \frac{1}{x+y}(1+y') = e^{y\ln x}(y'\ln x + \frac{y}{x}) \tag{1}$$

Now we substitute $x=1, y=0$ into the equation:

$$\frac{1}{0^2+1}(0+1y') + \frac{1}{1+0}(1+y') = e^{0\ln 1}(y'\ln 1 + \frac{0}{1})$$

Hence we get: $y' = -\dfrac{1}{2}$. And the line normal to that curve at $(1,0)$ is $(y-0) = -\dfrac{1}{y'}(x-1)$, i.e., $y = 2(x-1)$.

Now taking implicit differentiation on (1) again, we get:

$$\frac{((xy)^2+1)(y'+y'+xy'') - (y+xy')(2xy(y+xy'))}{((xy)^2+1)^2} + \frac{y''(x+y)-(1+y')^2}{(x+y)^2}$$
$$= e^{y\ln x}(y'\ln x + \frac{y}{x})^2 + e^{y\ln x}(y''\ln x + \frac{y'}{x} + \frac{y'x-y}{x^2})$$

Substitue $x=1, y=0, y' = -\dfrac{1}{2}$ into it, we get:

$$f'' = \frac{1}{8}$$

5. (14%) (a) Apply Generalized Mean Value Theorem to establish the inequalities

$$-\frac{1}{3} < \frac{\tan^{-1}x - x}{x^3} < \frac{-1}{3(1+x^2)},\ x > 0.\ (9\%)$$

(b) Use the result in (a) with $x = \dfrac{1}{\sqrt{3}}$ to find an interval that contains π. Use the midpoint of this interval to estimate π. Also find the error of this approximation. (5%)

Sol:

(a) By Generalized MVT, $x > 0$,

$$\frac{\tan^{-1} x - x}{x^3} = \frac{\frac{1}{1+c^2} - 1}{3c^2} = -\frac{1}{3(1+c^2)}$$

for some $c \in (0, x)$.

Since $1 < 1 + c^2 < 1 + x^2$,

$$-\frac{1}{3} < -\frac{1}{3(1+c^2)} < -\frac{1}{3(1+x^2)},$$

we have the inequalities.

(b) $\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$, thus

$$-\frac{1}{3} < \frac{\frac{\pi}{6} - \frac{1}{\sqrt{3}}}{\frac{1}{3\sqrt{3}}} < -\frac{1}{4}$$

$$\frac{16\sqrt{3}}{9} < \pi < \frac{11\sqrt{3}}{6}$$

Midpoint approximation gives

$$\pi \approx \frac{\sqrt{3}}{2}\left(\frac{16}{9} + \frac{11}{6}\right) = \frac{65}{36}\sqrt{3}$$

Error is less than or equal to half length of interval

$$\text{error} \leq \frac{\sqrt{3}}{2}\left(\frac{11}{6} - \frac{16}{9}\right) = \frac{\sqrt{3}}{36}$$

(Students can get 1% if error is estimated by total length of interval.)

6. (20%) Graph $y = f(x) = (x+2)e^{\frac{1}{x}}$. Be sure to write down the critical points, the intervals of monotonicity, the points of inflection, the intervals of concavity, and the asymptotes (if any).

Sol:

$y = f(x) = (x+2)e^{\frac{1}{x}}, x \neq 0$

$\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = 0$

$f'(x) = e^{\frac{1}{x}}(\frac{x^2 - x - 2}{x^2}) = e^{\frac{1}{x}}(\frac{(x-2)(x+1)}{x^2})$

critical points: $x = -1$, $x = 2$

intervals of increasing: $(-\infty, -1]$, $[2, \infty)$

intervals of decreasing: $[-1, 0)$, $(-, 2]$

$f''(x) = e^{\frac{1}{x}}(\frac{5x+2}{x^4})$

inflecion points: $x = -\frac{2}{5}$

intervals of concave up: $(-\frac{2}{5}, 0)$, $(0, \infty)$

intervals of concave down: $(-\infty, -\frac{2}{5})$

vertical asymptotes: $x = 0$,

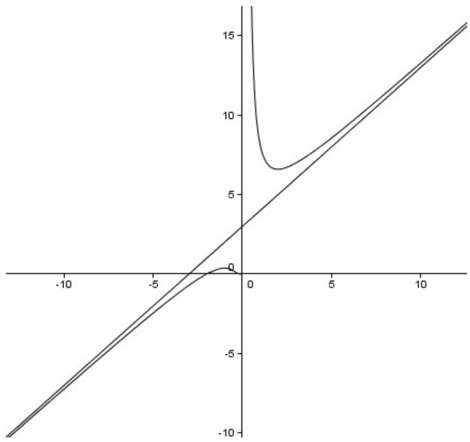
$\lim_{x\to\infty}(1+\frac{2}{x})e^{\frac{1}{x}} = 1$

$$\begin{aligned}
\lim_{x\to\infty}(x+2)e^{\frac{1}{x}} - x &= \lim_{x\to\infty} x[(1+\frac{2}{x})e^{\frac{1}{x}} - 1] \\
&= \lim_{x\to\infty} \frac{(1+\frac{2}{x})e^{\frac{1}{x}} - 1}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{-\frac{2}{x^2}e^{\frac{1}{x}} + (1+\frac{2}{x})(-\frac{1}{x^2})e^{\frac{1}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty}(2+1+\frac{2}{x})e^{\frac{1}{x}} = 3
\end{aligned}$$

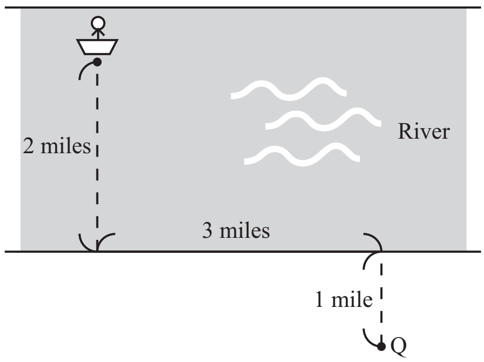
$\lim_{x\to-\infty}(1+\frac{2}{x})e^{\frac{1}{x}} = 1$

$$\begin{aligned}
\lim_{x\to-\infty}(x+2)e^{\frac{1}{x}} - x &= \lim_{x\to-\infty} x[(1+\frac{2}{x})e^{\frac{1}{x}} - 1] \\
&= \lim_{x\to-\infty} \frac{(1+\frac{2}{x})e^{\frac{1}{x}} - 1}{\frac{1}{x}} \\
&= \lim_{x\to-\infty} \frac{-\frac{2}{x^2}e^{\frac{1}{x}} + (1+\frac{2}{x})(-\frac{1}{x^2})e^{\frac{1}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to-\infty}(2+1+\frac{2}{x})e^{\frac{1}{x}} = 3
\end{aligned}$$

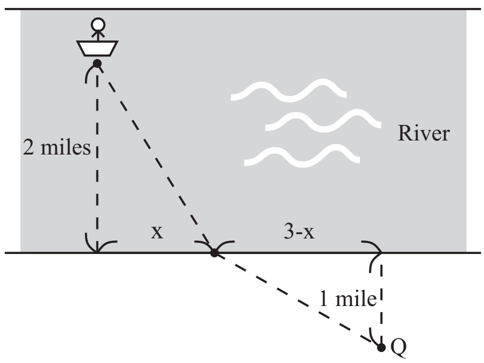
asymptotes: $y = x + 3$



7. (10%) A man is in a boat 2 miles away from the nearest point on the coast. He is going to a point Q, 3 miles down the coast and 1 mile in land. If he can row 2 miles per hour, and walk 4 miles per hour, toward what point on the coast should he row in order to reach Q in the least time?



Sol:



$$
\begin{aligned}
f(x) &= \frac{\sqrt{2^2+x^2}}{2} + \frac{\sqrt{1^2+(3-x)^2}}{4} \\
&= \frac{1}{4}(2\sqrt{4+x^2} + \sqrt{x^2-6x+10})
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \frac{1}{4}(\frac{2x}{\sqrt{4+x^2}} + \frac{2x-6}{2\sqrt{x^2-6x+10}}) \\
&= \frac{1}{4}(\frac{2x}{\sqrt{4+x^2}} + \frac{x-3}{\sqrt{x^2-6x+10}})
\end{aligned}
$$

$f'(x) = 0$

$\Rightarrow\ 2x\sqrt{x^2-6x+10} + (x-3)\sqrt{4+x^2} = 0$

$2x\sqrt{x^2-6x+10} = -(x-3)\sqrt{4+x^2}$

$4x^2(x^2-6x+10) = (x-3)^2(4+x^2)$

$4x^4 - 24x^3 + 40x^2 = 4x^2 - 24x + 36 + x^4 - 6x^3 + 9x^2$

$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$

$(x-1)(x^3 - 5x^2 + 4x - 12) = 0$

The process below is to prove the equation is above zero.

$x^3 - 4x^2 + 4x - 12 - x^2 = x(x-2)^2 + 12 - x^2 > 0$, for $0 \le x \le 3$

and $f'(1^-) < 0$, $f'(1) = 0$, $f'(1^+) > 0$

x=1 is the only critical point in [0,3]

$f(0) = \frac{\sqrt{4}}{2} + \frac{\sqrt{10}}{4} > 2$

$f(3) = \frac{\sqrt{13}}{2} + \frac{1}{4} > 2$

$f(1)$ is the least time:

$f(1) = \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{4} = \frac{3}{4}\sqrt{5} > 2$.