1001微甲01-05班期末考

1. (10%) Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} |\cos^2 x - 3\sin^2 x|\, dx$.

   Sol:

   We have known

$$\cos^2 x = \frac{1+\cos 2x}{2} \text{ (1 point) }, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

   Compute the integration

$$\int \cos^2 x - 3\sin^2 x \; dx = -x + \sin 2x + C \;, C \in \mathbb{R}$$

   In $[0, \frac{\pi}{2}]$, compute

$$\cos^2 x - 3\sin^2 x \geq 0 \Rightarrow 0 \leq x \leq \frac{\pi}{6}$$

   Then

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \left|\cos^2 x - 3\sin^2 x\right| \; dx \\
&= \int_0^{\frac{\pi}{6}} \cos^2 x - 3\sin^2 x \; dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 x - \cos^2 x \; dx \\
&= (-x + \sin 2x)\Big|_0^{\frac{\pi}{6}} + (x - \sin 2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{6} + \sqrt{3}
\end{aligned}$$

2. (15%) Compute $\displaystyle\int \frac{1}{e^{2x} + e^x + 1}\, dx$.

   Sol:

   Let $t = e^x \Rightarrow x = \ln t \Rightarrow dx = \dfrac{dt}{t}$

   Hence $\displaystyle\int \frac{1}{e^{2x} + e^x + 1} dx = \int \frac{1}{t^2 + t + 1} \frac{dt}{t}$

   Suppose that $\dfrac{1}{t(t^2+t+1)} = \dfrac{A}{t} + \dfrac{P(t)}{t^2+t+1}$

   $A(t^2 + t + 1) + P(t)t = 1$

   Let $\quad t = 0 \quad \Rightarrow A = 1$

   And then $P(t)t = -t^2 - t \Rightarrow P(t) = -t - 1$

   Consider the integral

$$
\begin{aligned}
\int \frac{1}{t^2+t+1}\frac{dt}{t} &= \int \frac{1}{t}dt + \int \frac{-t-1}{t^2+t+1}dt \\
&= \int \frac{1}{t}dt + (\frac{-1}{2})\int \frac{2t+1}{t^2+t+1}dt + \int \frac{\frac{-1}{2}}{t^2+t+1}dt \\
&= \ln|t| - \frac{1}{2}\ln|t^2+t+1| + \int \frac{\frac{-1}{2}}{t^2+t+1}dt
\end{aligned}
$$

Now we have to compute the third term

$$
\int \frac{\frac{-1}{2}}{t^2+t+1}dt = (\frac{-1}{2})\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{-1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}})
$$

So

$$
\begin{aligned}
&\ln|t| - \frac{1}{2}\ln|t^2+t+1| + \int \frac{\frac{-1}{2}}{t^2+t+1}dt \\
&= \ln|t| - \frac{1}{2}\ln|t^2+t+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}}) + C \\
&= x - \frac{1}{2}\ln|e^{2x}+e^x+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2e^x+1}{\sqrt{3}}) + C
\end{aligned}
$$

3. (15%) Let $I(p) = \int_0^\infty x^p e^{-x^2}\,dx$, where p is a real number.

   (a) Find p so that it converges. (8%)

   (b) Find $I(3)$. (7%)

Sol:

(a) $I(p) = \int_0^\infty x^p e^{-x^2} dx = \int_0^1 x^p e^{-x^2} dx + \int_1^\infty x^p e^{-x^2} dx$

Claim: $\int_1^\infty x^p e^{-x^2} dx$ converges for all $p \in \mathbb{R}$.

If $p \leq 0$, then $\int_1^\infty x^p e^{-x^2} dx \leq \int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = e < \infty$.

Suppose $p > 0$. Then there exists integers M_1 and M_2 such that $\frac{x^p}{e^x} \leq 1\ \forall x \geq M_1$ and $e^{-x^2} \leq e^{-2x}\ \forall x \geq M_2$. Let $M = \max\{M_1, M_2\}$. Then

$$
\begin{aligned}
\int_1^{\infty} x^p e^{-x^2} dx &= \int_1^{M} x^p e^{-x^2} dx + \int_M^{\infty} x^p e^{-x^2} dx \\
&\le \int_1^{M} x^p e^{-x^2} dx + \int_M^{\infty} x^p e^{-2x} dx \\
&= \int_1^{M} x^p e^{-x^2} dx + \int_M^{\infty} \frac{x^p}{e^x} e^{-x} dx \\
&\le \int_1^{M} x^p e^{-x^2} dx + \int_M^{\infty} e^{-x} dx < \infty
\end{aligned}
$$

Claim: $\int_0^1 x^p e^{-x^2} dx < \infty$ if and only if $p > -1$.

Since $e^{-1} \le e^{-x^2} \le e, \ \forall x \in [0, 1]$, we have

$$
e^{-1} \int_0^1 x^p dx \le \int_0^1 x^p e^{-x^2} dx \le e \int_0^1 x^p dx
$$

By the fact,

$$
\int_0^1 x^p dx \text{converges if and only if } p > -1,
$$

we have $\int_0^1 x^p e^{-x^2} dx < \infty$ if and only if $p > -1$.

(b) Let $u = x^2$. Then $du = 2xdx$. So

$$
\begin{aligned}
I(3) &= \int_0^{\infty} x^3 e^{-x^2} dx = \frac{1}{2} \int_0^{\infty} u e^{-u} du \\
&= \frac{-1}{2} [u(e^{-u}) \mid_0^{\infty} - \int_0^{\infty} e^{-u} du] \\
&= \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}
\end{aligned}
$$

4. (15%) Let the continuous function $f(x)$ be defined in $x > 0$, and satisfy $f(x) - x^4 = \int_1^{x^2} f(\sqrt{t}) t \, dt$. Find $f(x)$.

Sol:

$f(x)$ is a continuous function, therefore we differentiate both sides and get

$$
\begin{aligned}
f'(x) - 4x^3 &= \frac{d}{dx}\int_1^{x^2} f(\sqrt{t})tdt \\
&= \frac{d}{dy}\int_1^{y} f(\sqrt{t})tdt\frac{dy}{dx} \\
&= f\sqrt{y}y\frac{dx^2}{dx} \\
&= f(x)x^2 2x \\
&= 2x^3 f(x)
\end{aligned}
$$

Accordingly,

$$f'(x) - 2x^3 f(x) = 4x^3$$

Integrating factor:

$$e^{\int -2x^3 dx} = e^{\frac{-1}{2}x^4}$$

$$e^{\frac{-1}{2}x^4}[f'(x) - 2x^3 f(x)] = e^{\frac{-1}{2}x^4}4x^3$$

$$\frac{d}{dx}[e^{\frac{-1}{2}x^4}f(x)] = e^{\frac{-1}{2}x^4}4x^3$$

$$
\begin{aligned}
e^{\frac{-1}{2}x^4}f(x) &= \int e^{\frac{-1}{2}x^4}4x^3 dx \\
&\text{(Let } u = \frac{-1}{2}x^4, \text{ then } du = -2x^3dx, \ -2du = 4x^3dx) \\
&= \int e^u - 2du \\
&= -2e^u + C \\
&= -2e^{\frac{-1}{2}x^4} + C
\end{aligned}
$$

Consequently,

$$f(x) = -2 + Ce^{\frac{1}{2}x^4}$$

In addition,

$$f(1) - 1^4 = \int_1^1 f(\sqrt{t})tdt$$

We find

$$f(1) = 1$$

$$1 = f(1) = -2 + Ce^{\frac{1}{2}}$$

$$C = \frac{3}{e^{\frac{1}{2}}} = 3e^{\frac{-1}{2}}$$

Finally, we find:

$$f(x) = -2 + 3e^{\frac{-1}{2}+\frac{1}{2}x^4}$$

ps. Other methods (for example, seperation variables) solve the differential equation $f'(x) - 2x^3 f(x) = 4x^3$ are also permitted.

5. (15%) Consider the plane curve $3ay^2 = x(a-x)^2$ where $a > 0$ is a constant.

   (a) Find the arc length of the loop defined by the curve. (7%)

   (b) Find the surface area of the surface obtained by rotating the loop around x-axis. (8%)

Sol:

(a) Find the arc length

$3ay^2 = x(a-x)^2$

$\Longrightarrow 3a(2y)\frac{dy}{dx} = 2x(a-x)(-1) + (a-x)^2 \Longrightarrow \frac{dy}{dx} = \frac{(a-x)(a-3x)}{6ay}$

$$S = \int_0^a ds$$

$$= \int_0^a \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_0^a \sqrt{1 + \frac{(a-x)^2(a-3x)^2}{36a^2y^2}}dx$$

$$= \int_0^a \sqrt{1 + \frac{(a-x)^2(a-3x)^2}{3ay^2(12a)}}dx = \int_0^a \sqrt{1 + \frac{(a-x)^2(a-3x)^2}{x(a-x)^2(12a)}}dx$$

$$= \int_0^a \sqrt{\frac{(a+3x)^2}{12ax}}dx = \int_0^a \sqrt{\frac{1}{12a}}\frac{(a+3x)}{\sqrt{x}}dx$$

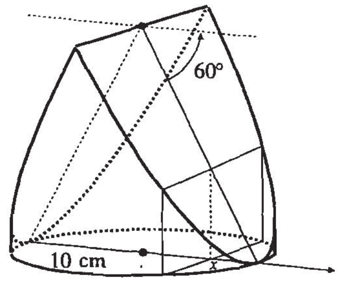
$$= \int_0^a \sqrt{\frac{1}{12a}}\frac{a}{\sqrt{x}}dx + \int_0^a \sqrt{\frac{1}{12a}}3x^{\frac{1}{2}}dx = \frac{a}{\sqrt{3}} + \frac{a}{\sqrt{3}} = \frac{2\sqrt{3}a}{3}$$

Length$=2 \times S = \frac{4\sqrt{3}}{3}a$

(b) Find the surface area

$$
\begin{aligned}
A &= \int_0^a 2\pi y ds \\
&= \int_0^a 2\pi\sqrt{1+(\frac{dy}{dx})^2} y dx = 2\pi\int_0^a \frac{1}{\sqrt{12a}}\frac{(a+3x)}{\sqrt{x}}\frac{\sqrt{x}(a-x)}{\sqrt{3}a}dx \\
&= 2\pi\int_0^a \frac{1}{6a}(a+3x)(a-x)dx = \frac{\pi}{3a}\int_0^a (a^2-ax+3ax-3x^2)dx \\
&= \frac{\pi}{3a}(a^3-\frac{1}{2}a^3+\frac{3}{2}a^3-a^3) = \frac{\pi}{3}a^2
\end{aligned}
$$

6. (15%) The solid in the figure is cut from a vertical cylinder of radius 10 cm by two planes making angles of 60° with the horizontal. Find its volume.



Sol:

The cross section perpendicular to the base of cylinder and parallel to the intersection of two planes is rectangle with width $\sqrt{10^2-x^2}$ and height $\sqrt{3}(10-x)$, where x is the distance to center of the cylinder, so the area of cross section is $2\sqrt{3}(10-x)\sqrt{10^2-x^2}$.

The volume is

$$
\begin{aligned}
&2\int_0^{10} 2\sqrt{3}(10-x)\sqrt{10^2-x^2}dx \\
&= 4\sqrt{3}\int_0^{\frac{\pi}{2}} 10(1-\sin u)10\sqrt{1-\sin^2 u}10\cos u du \\
&= 4000\sqrt{3}\int_0^{\frac{\pi}{2}} (1-\sin u)\cos^2 u du \\
&= 4000\sqrt{3}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2u}{2}du + 4000\sqrt{3}\int_1^0 v^2 dv \\
&= 4000\sqrt{3}\frac{\pi}{2}\frac{1}{2} + 4000\sqrt{3}\frac{1}{2}\frac{\sin\pi-\sin 0}{2} - 4000\sqrt{3}\frac{1}{3}(1^3-0^3) \\
&= 1000\sqrt{3}\pi - \frac{4000\sqrt{3}}{3}
\end{aligned}
$$

7. (15%) A plane curve is defined by $(x^2+y^2)^2 = 2xy$.

(a) Find its polar coordinate equation, and draw the graph. (5%)

(b) Find the area of the plane region bounded by the curve. (5%)

(c) Find the arc length of the curve. (5%)

Sol:

(a) $x = r\cos\theta, \quad y = r\sin\theta,$

$(r^2)^2 = 2r^2\cos\theta\sin\theta,$

$r^2 = \sin 2\theta,$

(b)

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r^2 d\theta \\ &= 2\times\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = 1\end{aligned}$$

(c)

$$\begin{aligned}\text{Arc length} &= \int \sqrt{(r'\theta)^2 + r^2} \\ &= \int \sqrt{\csc\theta} d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\csc\theta} d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{\csc\theta} d\theta\end{aligned}$$