Solutions of Homework #5

3. Consider $\alpha \in \mathbb{R}$. If $\alpha < 0 \Rightarrow \{x|f_n(x) \leq \alpha\} = \emptyset \Rightarrow \{x|f_n(x) \leq \alpha\}$ is measurable.
   If $\alpha \geq 0$, let $k = min\{y|y \in \mathbb{N} \cup \{0\}, \alpha \geq y\}$. Then
   $$\{x|f_n(x) \leq \alpha\} = \bigcup_{t=0}^{k} \bigcup_{x_i \in I, i=1,\ldots n-1} (0.x_1x_2\cdots x_{n-1}t, 0.x_1x_2\cdots x_{n-1}(t+1)],$$
   where $I = \{0, 1, \cdots, m-1\}$.
   Since $(0.x_1x_2\cdots x_{n-1}t, 0.x_1x_2\cdots x_{n-1}(t+1)]$ is an interval and therefore measurable, $\{x|f_n(x) \leq \alpha\}$ is also measurable.
   Therefore $f_n(x)$ is measurable , $\forall n$.

4. (a) $f(x) = sup_n f_n(x)$

   Claim : $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) = \{x|f(x) > \alpha\} = \bigcup_n f_n^{-1}((\alpha, \infty])$.
   pf : $\forall x \in f^{-1}((\alpha, \infty]) \Rightarrow f(x) > \alpha$.
   Since $f = sup_n f_n$, therefore
   $$\forall 0 < \varepsilon < f(x) - \alpha, \exists n_0 \in \mathbb{N} \text{ s.t.} f_{n_0}(x) + \varepsilon > f(x).$$
   Therefore,
   $$f_{n_0}(x) > f(x) - \varepsilon > f(x) - (f(x) - \alpha) = \alpha.$$
   $\therefore x \in f_{n_0}^{-1}((\alpha, \infty]) \Rightarrow x \in \bigcup_n f_n^{-1}((\alpha, \infty])$.
   Thus, $f^{-1}((\alpha, \infty]) \subset \bigcup_n f_n^{-1}((\alpha, \infty])$.
   On the other hand, $\forall x \in \bigcup_n f_n^{-1}((\alpha, \infty]) \Rightarrow x \in f_{n_0}^{-1}((\alpha, \infty])$, for some n_0,
   $\Rightarrow f(x) \geq f_{n_0}(x) > \alpha \Rightarrow x \in f^{-1}((\alpha, \infty])$ Hence we've proved this claim.

   Since we know $f_n's$ are measurable, $f_n^{-1}((\alpha, \infty])$ is measurable. And therefore $f^{-1}((\alpha, \infty]) = \bigcup_n f_n^{-1}((\alpha, \infty])$ is measurable. $\Rightarrow f$ is measurable.

   (b) (1) We have known that $inf_n f_n(x) = -sup_n(-f_n(x))$ $\therefore$ if $f_n(x)'s$ are measurable ,by (a), $\inf_n f_n(x)$ is also measurable.
   (2) Set $g(x) := \limsup_{n\to\infty} f_n(x) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k(x) =: \inf_{n\in\mathbb{N}} g_n(x)$.
   Here we define $g_n(x) = \sup_{k\geq n} f_k(x)$.
   Then by (a), we know g_n'sare measurable, and by(1), $g(x)$ is also measurable.

5. This assertion is NOT true!!
   Let $V \subset [0, 1]$ be a non-measurable set. Define
   $$f(x) = \begin{cases} 1, & x \in V \\ -1, & x \in [0,1] \setminus V \end{cases}$$
   Then $|f|(x) = 1$ on $[0, 1]$.
   Thus $|f|$ is measurable on ([0,1])
   But, $f^{-1}((0, \infty]) = V$ is not measurable, therefore f is not measurable!

6. Let V be the Vitali-type nonmeasurable set with $\lambda^*(V) = 1$.

(1) Suppose to the contrary that $\exists$ measurable set $A \subset V$ and $\lambda(A) > 0$.
Then let $A_1 = \{x - y | x, y \in A\}$ and $V_1 = \{x - y | x, y \in V\}$.
$\Rightarrow \exists \varepsilon \in \mathbb{R}$ such that $(-\varepsilon, \varepsilon) \subset A_1 \Rightarrow (-\varepsilon, \varepsilon) \subset V_1$.
Since $(-\varepsilon, \varepsilon)$ is an interval $\Rightarrow \exists k \in \mathbb{Q}$ such that $k \in (-\varepsilon, \varepsilon)$.
Then $k \in V_1$. Then we get a contradiction.

(2) Suppose to the contrary that $\exists$ a measurable set $B \subset V^c$ and $\lambda(B) > 0$.
$\Rightarrow [0, 1] \supset B^c \supset V \Rightarrow \lambda^*([0, 1]) \geq \lambda^*(B^c) \geq \lambda^*(V)$.
$\Rightarrow \lambda^*(B^c) = 1 \Rightarrow \lambda^*(B) = \lambda(B) = 0$. We get a contradiction!