Solutions of Homework #3

1. **Claim** : $\mathcal{S}_E$ is a σ-algebra on E.
   Suppose that $B \in \mathcal{S}_E \Rightarrow \exists A \in \mathcal{S}$ such that $B = A \cap E$.
   $\therefore B^c = A^c \cap E$.
   Since $\mathcal{S}$ is a σ-algebra $\Rightarrow A^c \in \mathcal{S} \Rightarrow B^c \in \mathcal{S}_E$.
   Suppose $B_i \in \mathcal{S}_E \Rightarrow \exists A_i \in \mathcal{S}$ such that $B_i = A_i \cap E$.
   $\therefore \bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty}(A_i \cap E) = (\bigcap_{i=1}^{\infty} A_i) \cap E$.
   Since $\mathcal{S}$ is a σ-algebra $\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{S} \Rightarrow \bigcap_{i=1}^{\infty} B_i \in \mathcal{S}_E$.

   Therefore $\mathcal{S}_E$ is a σ-algebra on E.

   Claim : ν is a measure on E.

   (1) : ν is well-defined.
   By the assumption, $\mu^*(X) = \mu^*(E)$, we have

   $$\mu^*(X) = \mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$$

   Since $A = A \cap X \supseteq A \cap E \Rightarrow \mu^*(A) = \mu^*(A \cap X) \geq \mu^*(A \cap E)$, and $A^c = A^c \cap X supseteq A^c \cap E \Rightarrow \mu^*(A^c) = \mu^*(A^c \cap X) \geq \mu^*(A^c \cap E)$. $\because \mu^*(X)$ is finite and $\mu^*(X) = \mu^*(E)$, we have

   $$\mu^*(A) = \mu^*(A \cap E) \text{ and } \mu^*(A^c) = \mu^*(A^c \cap E).$$

   Similarly, $\mu^*(B) = \mu^*(B \cap E)$. Therefore

   $$\mu(A) = \mu^*(A) = \mu^*(A \cap E) = \mu^*(B \cap E) = \mu^*(B) = \mu(B).$$

   (2) :$\nu(\emptyset) = 0$
   $\nu(\emptyset) = \nu(\emptyset \cap E) = \nu(\emptyset) = 0$.

   (3) :$A \cap E \supseteq B \cap E \Rightarrow \nu(A \cap E) \geq \nu(B \cap E)$.
   Since ν is well-defined, $\therefore (A \cap E) \supseteq (B \cap E) \Rightarrow \mu(A) \geq \mu(B)$.
   Terefore $\nu(A \cap E) = \mu(A) \geq \mu(B) = \nu(B \cap E)$.

   (4) :Countable additivity.
   Let $B_i = A_i \cap E \in \mathcal{S}_E$, $A_i \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$, $\forall i \neq j$.
   Let $C_1 = A_1$, $C_i = A_i \setminus (\bigcup_{j=1}^{i-1})$, $\forall i \geq 2$.
   $\therefore C_i \in \mathcal{S}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} C_i$, C_i disjoint, and

   $$C_i \cap E = B_i = A_i \cap E.$$

   Therefore

   $$\begin{aligned}
   \nu(\bigcup_{i=1}^{\infty} B_i) &= nu(\bigcup_{i=1}^{\infty}(A_i \cap E)) = \mu(\bigcup_{i=1}^{\infty} A_i) \\
   &= \mu(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu(C_i) \\
   &= \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \nu(A_i \cap E) = \sum_{i=1}^{\infty} \nu(B_i)
   \end{aligned}$$

2. (i) :

Suppose $B \in \mathcal{A}$, $\therefore B = ((a_1, b_1] \cup \ldots \cup (a_n, b_n]) \cap X$, where

$$-\infty \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n \leq \infty.$$

$\therefore B^c$ on $X = ((-\infty, a_1] \cup (\bigcup_{i=1}^{n-1}(b_i, a_{i+1}]) \cup (a_n, \infty]) \cap X$

$\therefore B^c \in \mathcal{A}$.

Suppose $C \in \mathcal{A}$, $\therefore C = ((c_1, d_1] \cup \ldots \cup (c_m, d_m]) \cap X$, where

$$-\infty \leq c_1 \leq d_1 \leq c_2 \leq d_2 \leq \ldots \leq c_m \leq d_m \leq \infty.$$

If $(a, b] \cap (c, d] \neq \emptyset$, then $(a, b] \cup (c, d] = (e, f]$, for some e, f. Therefore,

$$\begin{aligned} B \cup C &= (\bigcup_{i=1}^{n}(a_i, b_i] \cap X) \cup (\bigcup_{j=1}^{m}(c_j, d_j] \cap X) \\ &= (\bigcup_{i=1}^{n}(a_i, b_i] \cup \bigcup_{j=1}^{m}(c_j, d_j]) \cap X = (\bigcup_{k=1}^{l}(e_k, f_k]) \cap X, \text{ for some } e_k, f_k \end{aligned}$$

$\therefore B \cup C \in \mathcal{A}$ and $\mathcal{A}$ is an algebra.

(ii) :

Since 2^X is a σ-algebra and $\mathcal{A} \subset 2^X \Rightarrow \sigma(\mathcal{A}) \subset 2^X$.

Consider $x \in \mathbb{Q}$, since $\{x\} = \bigcap_{n=1}^{\infty}((x - \frac{1}{n}, x] \cap X) \Rightarrow \{x\} \in \sigma(\mathcal{A})$.

Suppose $S \in 2^X \Rightarrow S = \{x_1, x_2, \ldots\}$ by the fact that X is countable.

$\therefore S = \bigcup_{i=1}^{\infty}\{x_i\}$, $x_i \in \mathbb{Q}$, $\forall i$.

Since $\{x_i\} \in \sigma(\mathcal{A}) \Rightarrow S \in \sigma(\mathcal{A}) \Rightarrow 2^X \subset \sigma(\mathcal{A})$.

Hence $2^X = \sigma(\mathcal{A})$.

(iii) :

Let

$$\mu_1(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ \infty, & \text{otherwise} \end{cases}$$

and $\mu_2(S) =$ the cardinality of S.

Suppose $A \in \mathcal{A}$.

If $A = \emptyset$, $\mu_1(\emptyset) = \mu_2(\emptyset) = \mu(\emptyset) = 0$.

If $A \neq \emptyset \Rightarrow A = \{x_1, x_2, \ldots\} \Rightarrow \mu_1(A) = \mu_2(A) = \mu(A) = \infty$.

Therefore μ_1 and μ_2 are the extension of μ to $\sigma(\mathcal{A})$.

But $\mu_1(\{x\}) = \infty \neq 1 = \mu(\{x\})$.

Hence the extension of μ to $\sigma(\mathcal{A})$ is not unique.

3. Note :$\mathcal{S}$ is a σ-algebra ,and (X, S, μ) is a measure space, but may not be complete.

(i) By definition, $\mathcal{N}(\mu) = \{E \subset X \| \mu^*(E) = 0\}$.

1^0 Let $\mathcal{A} = \{E \subset X \| E \subset N$ for some $N \in \mathcal{S}$ with $\mu(N) = 0\}$.
$\forall E \in \mathcal{A}, \exists N \in \mathcal{S}$ with $E \subset N$ such that $\mu(N) = 0$.
$\Rightarrow 0 \leq \mu^*(E) \leq \mu^*(N) = \mu(N) = 0$.
$\therefore \mu^*(E) = 0 \Rightarrow E \in \mathcal{N}(\mu)$.
On the other hand, given $E \in \mathcal{N}(\mu)$, $\exists$ measurable cover $B \in \mathcal{S}$ of E such that $E \subset B$ and $\mu(B) = \mu^*(E) = 0$.
$\therefore E \in \mathcal{A}$.
Hence $\mathcal{A} = \mathcal{N}(\mu)$.

2^0 By definition, $S \vee \mathcal{N}(\mu) =$ the σ-algebra generated by $S \cup \mathcal{N}(\mu)$.
Let $\mathcal{B} = \{E \cup F \| E \in \mathcal{S}$ and $F \in \mathcal{N}(\mu)\}$.
Obviously, $\mathcal{B} \subset \S \vee \mathcal{N}(\mu)$.
Claim:$\mathcal{B}$ is a σ-algebra containing $S \cup \mathcal{N}(\mu)$.
pf:Since $\emptyset \in \mathcal{S} \cap \mathcal{N}(\mu)$, $\forall E \in \mathcal{S}, F \in \mathcal{N}(\mu)$
$\Rightarrow E = E \cup \emptyset \in \mathcal{B}$ and $F = \emptyset \cup F \in \mathcal{B}$.
$\therefore \mathcal{S} \cup \mathcal{N}(\mu) \subset \mathcal{B}$. Now, we're going to prove that $\mathcal{B}$ is a σ-algebra.
First, it's easy to see $\emptyset \in \mathcal{B}$.
Second, given $\{B_n\}_{n=1}^{\infty} \subset \mathcal{B}$
$\Rightarrow B_n = E_n \cup F_n$, for some $E_n \in \mathcal{S}$ and $F_n \in \mathcal{N}(\mu)$. Thus,

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty}(E_n \cup F_n) = (\bigcup_{n=1}^{\infty} E_n) \cup (\bigcup_{n=1}^{\infty} F_n). \tag{1}$$

Since $\mathcal{S}$ is a σ-algebra, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$.
Now we can choose a measurable cover N_n of F_n, form the $\underline{problem4}$, we know that $\bigcup_{n=1}^{\infty} N_n$ is also a measurable cover of $\bigcup_{n=1}^{\infty} F_n$. Therefore

$$\mu^*(\bigcup_{n=1}^{\infty} F_n) \leq \mu(\bigcup_{n=1}^{\infty} N_n) \leq \sum_{n=1}^{\infty} \mu(N_n) = 0$$

$\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} F_n) = 0 \Rightarrow \bigcup_{n=1}^{\infty} F_n \in \mathcal{N}(\mu)$ (by 1^0). Hence by (1), $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.
Third, given $B \in \mathcal{B} \Rightarrow B = E \cup F$, for some $E \in \mathcal{S}$ and $F \in \mathcal{N}(\mu)$.
$\Rightarrow \exists N \in \mathcal{S}$ such that $F \subset N$ and $\mu(N) = 0$.
$\Rightarrow B^c = (E \cup F)^c = E^c \cap F^c = (E^c \cap N^c) \cup (N \setminus F)$.
Since $(N \setminus F) \subset N$ and $\mu(N) = 0$, $(N \setminus F) \in \mathcal{N}(\mu)$.
Therefore $B^c \in \mathcal{B}$, since $E, N \in \mathcal{S}$, and we complete the proofof the claim.

(ii) : $\bar{\mu}$ is the completion of μ.
$\Rightarrow Dom(\bar{\mu}) = \mathcal{S} \vee \mathcal{N}(\mu)$.

($\Rightarrow$) : $\forall E \subset Dom(\bar{\mu})$.
$\Rightarrow E \in \mathcal{S} \vee \mathcal{N}(\mu) = \mathcal{S}_{\mu^*} = \{E \subset X \| E \triangle B \in \mathcal{N}(\mu)$, for some $B \in \mathcal{S}\}$.
$\Rightarrow \exists B \in \mathcal{S}$ s.t. $E \triangle B \in \mathcal{N}(\mu)$.

$\Rightarrow$ Let $C = E \cup B$, $A = E \cap B$.
Then $A \subset E \subset C$ and $A, C \in Dom(\bar{\mu})$, and therefore we have

$$0 \leq \bar{\mu}(C \setminus A) = \bar{\mu}(C \setminus E) + \bar{\mu}(E \setminus A)$$
$$\leq 2\bar{\mu}(B \bigtriangleup E) = 2\mu(B \bigtriangleup E) = 0.$$

Therefore $\mu(C \setminus A) = \bar{\mu}(C \setminus A) = 0$

($\Leftarrow$) : For any E with the following property,
$\exists A, C \in Dom(\bar{\mu})$ with $A \subset E \subset C$ s.t. $\mu(C \setminus A) = 0$.
Since $(E \setminus A)$ $(C \setminus A)$ and $\mu(C \setminus A) = 0$, we have $E \setminus A \in \mathcal{N}(\mu)$.
Consider $E = A \cup (E \setminus A)$.
Since $A \in \mathcal{S}$ and $(E \setminus A) \in \mathcal{N}(\mu)$, we have $E \in \mathcal{S} \vee \mathcal{N}(\mu) = Dom(\bar{\mu})$.

4. (i) : Let C_j be a measurable cover of A_j.
Then C_j is a μ^*-measurable set and

$$\mu^*(C_j) = \mu^*(A_j) \text{ with } C_j \supset A_j.$$

Now, let B be a measurable cover of $\bigcup_j A_j$, then

$$\mu^*(\bigcup_j A_j) = \mu^*(B) \text{ and } \bigcup_j A_j \subset B.$$

Set $\tilde{C}_j = B \cap C_j$. Thus $A_j \subset \tilde{C}_j$. Moreover, we have

$$\mu^*(A_j) \leq \mu^*(\tilde{C}_j) \leq \mu^*(C_j).$$

$\therefore \tilde{C}_j$ is a measurable cover of A_j.
Consider $\mu^*(C_j \bigtriangleup \tilde{C}_j) = \mu^*(C_j \setminus \tilde{C}_j)$.

(a) : If $\mu^*(\tilde{C}_j) = \infty$ for some j
Then $\mu^*(A_j) = \mu^*(C_j) = \mu^*(\tilde{C}_j) = \infty$
$\Rightarrow \mu^*(\bigcup_j C_j) = \mu^*(\bigcup_j A_j) = \infty$ and $(\bigcup_j C_j) \supset (\bigcup_j A_j)$
Therefore $\bigcup_j C_j$ is a measurable cover of $\bigcup_j A_j$.

(b) : If $\mu^*(\tilde{C}_j) < \infty$, $\forall j$.
Then $\mu^*(C_j \bigtriangleup \tilde{C}_j) = \mu^*(C_j) - \mu^*(\tilde{C}_j) = \mu^*(A_j) - \mu^*(A_j) = 0$.
By <u>*homework#2*</u>,

$$\mu^*(\bigcup_j C_j) = \mu^*(\tilde{C}_j) \leq \mu^*(B) = \mu^*(\bigcup_j A_j)$$

Since $(\bigcup_j A_j) \subset (\bigcup_j C_j)$, we have $\mu(\bigcup_j C_j) = \mu^*(\bigcup_j A_j)$.
Therefore $\bigcup_j C_j$ is also a measurable cover of $\bigcup_j A_j$

(ii) There are many examples! For example:

(1) $X = \mathbb{R}$, $\mathcal{S} = \{A \subset X | A \text{ is countable}\} \cup \{A \subset X | A^c \text{ is countable}\}$.
Write $\mathcal{A}_1 = \{A \subset X | A \text{ is countable}\}$ and $\mathcal{A}_2 = \{A \subset X | A^c \text{ is countable}\}$.
Define

$$m(A) = \begin{cases} 0, & \text{if } A \in \mathcal{A}_1 \\ 1, & \text{if } A \in \mathcal{A}_2 \end{cases}$$

Let $A_1 = (-\infty, 0)$, $A_2 = (0, \infty) \Rightarrow \mu^*(A_1) = \mu^*(A_2) = 1$.
Let $C_1 = C_2 = \mathbb{R}$, then $\mu^*(A_i) = \mu(C_i) = 1, i = 1, 2$.
Thus, $\mu^*(A_1 \cap A_2) = 0 \neq 1 = \mu(\mathbb{R}) = \mu(C_1 \cap C_2)$.

(2) $X = \mathbb{R}$ and m is the counting measure.
Set $A_1 = (0, 1)$ and $A_2 = (1, 2)$ and $C_1 = [0, 1]$, $C_2 = [1, 2]$.
Then $\mu^*(A_i) = \mu^*(C_i) = \infty$, $i = 1, 2$ and
$\mu^*(A_1 \cap A_2) = 0 \neq 1 = \mu^*(C_1 \cap C_2)$.