Solutions of Homework #2

1. $\mathcal{A} = \{A \subset X | A \text{ or } A^c \text{ is finite}\}$.

(a)
($\mathcal{A}$ is an algebra)
(i) $\because \emptyset = X^c$ is finite, $\therefore \emptyset, X \in \mathcal{A}$.
(ii) Let $A \in \mathcal{A}$.
If A is finite $\Rightarrow (A^c)^c = A$ is finite $\Rightarrow A^c \in \mathcal{A}$.
If A^c is finite $\Rightarrow A^c \in \mathcal{A}$.
(iii) Let $\{A_n\}_{n=1}^k \subset \mathcal{A}$.
If $A_n's$ are finite. $\Rightarrow \bigcup_{n=1}^k A_n$ is finite. $\therefore \bigcup_{n=1}^k A_n \in \mathcal{A}$.
If there is $A_{n_0} \in \mathcal{A}$ such that $A_{n_0}^c$ is finite.
$\Rightarrow (\bigcup_{n=1}^k A_n)^c = \bigcap_{n=1}^k A_n^c \subset A_{n_0}^c$, $\therefore (\bigcup_{n=1}^k A_n)^c$ is finite.
$\therefore \bigcup_{n=1}^k A_n \in \mathcal{A}$.
(A is not a σ-algebra)
Let $X = \{x_n\}_{n=1}^\infty$ and $A_k = \{x_{2k}\}$ for $k = 1, 2, \ldots$.
Then $\bigcup_{k=1}^\infty A_k$ is infinite and $(\bigcup_{k=1}^\infty A_k)^c = \{x_{2k-1}\}_{k=1}^\infty$ is also infinite. Therefore $\bigcup_{k=1}^\infty A_k$ doesn't belong to $\mathcal{A}$ and $\mathcal{A}$ is not a σ-algebra.

(b) Let $\{A_n\}_{n=1}^k \subset \mathcal{A}$ be such that $A_n \cap A_m = \emptyset, \forall m \neq n$. Then
(i) If $A_n's$ are finite, then $\bigcup_{n=1}^k A_n$ is finite.
$\Rightarrow m(\bigcup_{n=1}^k A_n) = 0 = \sum_{n=1}^k m(A_n)$.
(ii) If there is A_{n_0} such that $(A_{n_0})^c$ is finite, then $m(A_{n_0}) = 1$.
$\Rightarrow (\bigcup_{n=1}^k A_n)^c = \bigcap_{n=1}^k (A_n)^c \subset A_{n_0}$ is finite.
$\therefore m(\bigcup_{n=1}^k A_n) = 1$.
On the other hand, $\because A_n \cap A_m = \emptyset, \forall m \neq n$
$\therefore \bigcup_{n \neq n_0} A_n \subset (A_{n_0})^c$.
$\therefore \bigcup_{n \neq n_0} A_n$ is finite. $\Rightarrow m(\bigcup_{n \neq n_0} A_n) = 0 = \sum_{n \neq n_0} m(A_n)$.
Hence $m(\bigcup_{n=1}^k A_n) = 1 = m(A_{n_0}) + 0 = m(A_{n_0}) + \bigcup_{n \neq n_0} m(A_n) = \sum_{n=1}^k m(A_n)$.

(c) If X is <u>*uncountable*</u>, then m can be extended to a countably additive measure on a σ-algebra.
Claim: m is countably additive on $\mathcal{A}$.
pf: Let $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ be such that

$$A_n \cap A_m = \emptyset, \forall n \neq m \text{ and } \bigcup_{n=1}^\infty A_n \in \mathcal{A}.$$

(i) If A_n is finite, $\forall n$ and $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$, then we will claim $\bigcup_{n=1}^\infty A_n$ is finite.
if not, $(\bigcup_{n=1}^\infty A_n)^c$ is finite.
Because X is uncountable, we know that $\bigcup_{n=1}^\infty A_n$ is also uncountable.
Then we get a contradiction by the fact that the $A_n's$ are finite.

(ii) If there is A_{n_0} such that $(A_{n_0})^c$ is finite, then $(\bigcup_{n=1}^{\infty} A_n)^c \subset (A_{n_0})^c$ is finite.
$\therefore m(\bigcup_{n=1}^{\infty} A_n) = 1$.
On the other hand, $\because A_n \cap A_{n_0} = \emptyset, \forall n \neq n_0$.
$\therefore \bigcup_{n \neq n_0} A_n \subset A_{n_0}^c \Rightarrow \bigcup_{n \neq n_0} A_n$ is finite.
$\therefore m(\bigcup_{n \neq n_0} A_n) = 0 = \bigcup_{n \neq n_0} m(A_n)$.
So $m(\bigcup_{n=1}^{\infty} A_n) = 1 = m(A_{n_0}) = \sum_{n=1}^{\infty} m(A_n)$.

Since m is countably additive on the algebra $\mathcal{A}$, m can be extended to a countably additive measure on a σ-algebra.
If fact, if we set

$$S = \{A \subset X | A \text{ or } A^c \text{ is countable } \} \text{ and}$$

$$\tilde{m}(A) = \begin{cases} 1, & \text{if } A^c \text{ is countable} \\ 0, & \text{if } A \text{ is countable} \end{cases}$$

Then we can check that S is a σ-algebra and $\tilde{m}$ is a measure on S such that $\tilde{m} = m$ on $\mathcal{A}$.

2.

$$\begin{aligned} \mu^*(\bigcup_j A_j) &\leq \mu^*(\bigcup_j A_j \cup \bigcup_j B_j) \\ &= \mu^*(\bigcup_j (A_j \cup B_j)) = \mu^*(\bigcup_j ((A_j \triangle B_j) \cup (A_j \cap B_j))) \\ &= \mu^*((\bigcup_j (A_j \triangle B_j)) \cup (\bigcup_j (A_j \cap B_j))) \\ &\leq \mu^*((\bigcup_j (A_j \triangle B_j)) + \mu^*(\bigcup_j (A_j \cap B_j))) \\ &\leq \sum_j \mu^*(A_j \triangle B_j) + \mu^*((\bigcup_j A_j) \cap (\bigcup_j B_j)) = \mu^*((\bigcup_j A_j) \cap (\bigcup_j B_j)) \\ &\leq \mu^*(\bigcup_j A_j) \end{aligned}$$

Therefore $\mu^*(\bigcup_j A_j) = \mu^*((\bigcup_j A_j) \cap (\bigcup_j B_j))$.
Similarly, $\mu^*(\bigcup_j B_j) = \mu^*((\bigcup_j A_j) \cap (\bigcup_j B_j))$.
Hence, $\mu^*(\bigcup_j A_j) = \mu^*(\bigcup_j B_j)$.

3. Let F be a closed subset of X. We want to show that $\forall A \in 2^X$

$$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F).$$

Consider $A_n = \{x \in A \setminus F | d(x, F) \geq \frac{1}{n}\}$.
Since $\rho(A_n, A \cap F) = inf\{d(x, y) | x \in A_n, y \in A \cap F\} > 0$, by assumption we have

$$\mu^*(A_n \cup (A \cap F)) = \mu^*(A_n) + \mu^*(A \cap F)$$

Since $A_n \cup (A \cap F) \subset A$, therefore

$$\mu^*(A_n) + \mu^*(A \cap F) \leq \mu^*(A) \tag{1}$$

Claim(1): $\bigcup_{n=1}^{\infty} A_n = A \setminus F$.

Obviously, $\bigcup_{n=1}^{\infty} A_n \subset A \setminus F$.

If not, $\exists x \in (A \setminus F) \setminus \bigcup_{n=1}^{\infty} A_n$. $\Rightarrow d(x,F) < \frac{1}{n}, \forall n$.

Therefore $d(x,F) = 0$. Since F is closed, we have $x \in F$, and therefore we get a contradiction.

If we can show $\mu^*(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu^*(A_n)$, then let $n \to \infty$ in (1), we have

$$\mu^*(A \setminus F) + \mu^*(A \cap F) \leq \mu^*(A)$$

Since the opposite inequality is always true, F is μ^*-measurable.

Claim(2): $\mu^*(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu^*(A_n)$

There are two proof of this claim:

pf(a): We will show the general case, that is,

$$\text{If } B_n \bigcup_{n=1}^{\infty}, \text{ then } \mu^*(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu^*(B_n)$$

<u>*step*1</u> : If $\{B_n\}$ are μ^*-measurable, then the equality holds.(We have showed this in the class.)

<u>*step*2</u> : For each n, we can choose a measurable cover $\tilde{C}_n$ of B_n.

Let $C_n = \bigcap_{k=n}^{\infty} \tilde{C}_n$, then $C_n \nearrow$ and measurable. Moreover $\tilde{C}_k \supset B_k \supset B_n, \forall k \geq n$. Therefore $B_n \subset C_n$.

From the following inequality,

$$\mu^*(B_n) \leq \mu^*(C_n) \leq \mu^*(\tilde{C}_n) = \mu^*(B_n)$$

we have $\mu^*(C_n) = \mu^*(B_n)$. ie. C_n is also a measurable cover of B_n.

By problem 4 <u>*of Homework#*3</u>, $\bigcup_{n=1}^{\infty} C_n$ is a measurable cover of $\bigcup_{n=1}^{\infty} B_n$.

Then by <u>*step*1</u>, $\mu^*(\bigcup_{n=1}^{\infty} C_n) = \lim_{n\to\infty} \mu^*(C_n)$.

Hence $\mu^*(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu^*(B_n)$.

By Taking $A_n = B_n$, we proves this claim.

pf(b): (多數同學的證明)

If $\mu^*(A) = \infty$, then $\mu^*(A \cap F) + \mu^*(A \setminus F) \geq \mu^*(A) = \infty$.

Therefore $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F)$.

If $\mu^*(A) < \infty$.

Let $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$. Then

$$\rho(B_{2k}, B_{(2(k+1))}), \rho(B_{2k-1}, B_{2k+1}) > 0, \forall k \in \mathbb{N}$$

By assumption,

$$\mu^*(\bigcup_{k=1}^{\infty} B_{2k}) = \sum_{k=1}^{\infty} \mu^*(B_{2k}) \mu^*(\bigcup_{k=1}^{\infty} B_{2k-1}) = \sum_{k=1}^{\infty} \mu^*(B_{2k-1})$$

Since $\bigcup_{k=1}^{\infty} B_{2k}, \bigcup_{k=1}^{\infty} B_{2k-1} \subset A$, we have

$$\sum_{k=1}^{\infty} \mu^*(B_{2k}), \sum_{k=1}^{\infty} \mu^*(B_{2k-1}) < \infty.$$

Therefore $\forall \varepsilon > 0, \exists$ large integer k_0 such that

$$\sum_{k=k_0}^{\infty} \mu^*(B_{2k}), \sum_{k=k_0}^{\infty} \mu^*(B_{2k-1}) < \frac{\varepsilon}{2}.$$

Hence $\forall 2n > k_0$,

$$\begin{aligned} \mu^*(\bigcup_{n=1}^{\infty} A_n) &\leq \mu^*(A_{2n}) + \sum_{k=n+1}^{\infty} \mu^*(B_{2k}) + \sum_{k=n+1}^{\infty} \mu^*(B_{2k-1}) \\ &< \mu^*(A_{2n}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu^*(A_{2n}) + \varepsilon. \end{aligned}$$

Therefore $sup_n \mu^*(A_n) \geq \mu^*(\bigcup_{n=1}^{\infty} A_n)$.
$\because \mu^*(A_n) \nearrow$ and $\mu^*(A_n) \leq \mu^*(A) < \infty$,
$\therefore \lim_{n\to\infty} \mu^*(A_n) = sup_n \mu^*(A_n)$.
$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \lim_{n\to\infty} \mu^*(A_n)$. On the other hand, $\because A_n \subset \bigcup_{n=1}^{\infty} A_n$, the opposite inequality is also true.

4. Let $E \subset X$.
If $\nu^*(E) = \infty \Rightarrow \nu^*(E) \geq \mu^*(E)$.
If $\nu^*(E) < \infty$, given $\varepsilon > 0$, $\exists A_i \in \mathcal{A}$ such that

$$E \subset \bigcup_i A_i \text{ and } \sum_i \nu(A_i) \leq \nu^*(E) + \varepsilon.$$

Since $\mu = \nu$ on $\mathcal{A}$, we have $\sum_i \nu(A_i) = \sum_i \mu(A_i) \geq \mu^*(E)$. $\Rightarrow \mu^*(E) \leq \nu^*(E) + \varepsilon$, $\forall \varepsilon$.
$\Rightarrow \mu^*(E) \leq \nu^*(E)$, $\forall E$.

$\because \nu$ is σ-finite, countably additive, $\sigma(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$ and μ is the extension of ν. $\therefore \mu$ is the measure defined on $\sigma(\mathcal{A})$ and $\mu = \nu^*$ on $\sigma(\mathcal{A})$.

Let $E \subset X$.
If $\mu^*(E) = \infty \Rightarrow \mu^*(E) \geq \nu^*(E)$.
If $\mu^*(E) < \infty$, given $\varepsilon > 0$, $\exists B_i \in \sigma(\mathcal{A})$ such that

$$E \subset \bigcup_i B_i \text{ and } \sum_i \mu(B_i) \leq \mu^*(E) + \varepsilon.$$

Since $\mu = \nu^*$ on $\sigma(\mathcal{A})$, we have $\sum_i \mu(B_i) = \sum_i \nu^*(B_i) \leq \nu^*(E)$.
$\Rightarrow \nu^*(E) \leq \mu^*(E) + \varepsilon$, $\forall \varepsilon$.
$\Rightarrow \nu^*(E) \leq \mu^*(E)$, $\forall E$.
Hence $\mu^* = \nu^*$.