Solutions of Homework #1

1. By the assumption, $\mathcal{S} = \sigma(\mathcal{A})$ = the σ-algebra generated by $\mathcal{A}$.
   Let $F = \{\mathcal{E} \subset \mathcal{A} | \mathcal{E}$ has countable elements$\}$. We want to show that $\mathcal{S} = \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$.

   1^o claim:$\bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$ is also a σ-algebra containing $\mathcal{A}$.
   pf:$\bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E}) \supset \mathcal{A}$ is obviously.

   (i) $\forall \{E_n\}_{n=1}^{\infty} \subset \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$,
   there exists $\mathcal{E}_n \in F$ such that $E_n \in \mathcal{E}_n$, $\forall n$. Since $\mathcal{E}_n$ has countable elements, $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ has countable elements.
   Let $\bigcup_{n=1}^{\infty} \mathcal{E}_n = \mathcal{E}_0 \in F$, we have $E_n \in \mathcal{E}_0$, $\forall n$.
   Since $\sigma(\mathcal{E}_0)$ is a σ-algebra, then $\bigcup_{n=1}^{\infty} E_n \in \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$.

   (ii) Given $E_1,\ E_2 \in \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$. Similarly to (i), there exists $\mathcal{E}_1$ such that $E_1,\ E_2 \in \sigma(\mathcal{E}_1)$. Therefore,$E_1 \setminus E_2 \in \sigma(\mathcal{E}_1) \subset \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$.
   Moreover, $\emptyset,\ X \in \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$. Therefore by (i) and (ii), $\bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$ is a σ-algebra containing $\mathcal{A}$.

   Since $\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$, by the claim, $\mathcal{S} \subset \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E})$

   2^o On the other hand, for each $\mathcal{E} \in F$, $\mathcal{E} \subset \mathcal{A} \Rightarrow \sigma(\mathcal{E}) \subset \sigma(\mathcal{A}) = \mathcal{S}$, $\forall \mathcal{E} \in F$.
   $\Rightarrow \bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E}) \subset \mathcal{S}$.

   By 1^o and 2^o, $\bigcup_{\mathcal{E}\in F} \sigma(\mathcal{E}) = \mathcal{S}$

2. Since $\mathcal{A} \subset 2^X$ is an algebra, $\therefore X \in \mathcal{A}$.
   Let $\mathcal{M}(\mu^*)$ = the collection of μ^*-measurable sets.

   (i) Given $E \subset X$, we always can find $A_n \subset \mathcal{A}$ such that $E \subset \bigcup_{n=1}^{\infty} A_n$.($\because X \in \mathcal{A}$)
   Define

$$\mu^*(E) = inf\{\sum_{n=1}^{\infty} \mu(A_n) | E \subset \bigcup_{n=1}^{\infty} A_n, A_n \in \mathcal{A}\}.$$

   Therefore $\forall \varepsilon > 0$, $\exists \{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ with $E \subset \bigcup_{n=1}^{\infty} A_n$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) < \mu^*(E) + \varepsilon$$

   Let $A = \bigcup_{n=1}^{\infty} A_n$. $\Rightarrow A \in \mathcal{A}_{\sigma}$.

$$\begin{aligned} \mu^*(A) &= inf\{\sum_{n=1}^{\infty} \mu(B_n) | A \subset \bigcup_{n=1}^{\infty} B_n, B_n \in \mathcal{A}\} \\ &\leq \sum_{n=1}^{\infty} \mu(A_n) \\ &\leq \mu^*(E) + \varepsilon \end{aligned}$$

(ii) $\mu^*(E) < \infty$.

(⇒) Suppose that E is μ^*-measurable.
Given $F \subset X$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E) \tag{1}$$

By (i),we have $\forall n \in \mathbb{N}$, $\exists A_n \in \mathcal{A}_\sigma$ such that

$$\mu^*(A_n) \le \mu^*(E) + \frac{1}{n}$$

Let $B = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}_{\sigma\delta} \Rightarrow E \subset B$. Since $B \subset A_n$, $\forall n$, we have

$$\mu^*(B) \le \mu^*(A_n) \le \mu^*(E) + \frac{1}{n}, \forall n.$$

$\Rightarrow \mu^*(B) \le \mu^*(E)$.
By $E \subset B$, $\mu^*(B) = \mu^*(E)$. By (1), we have

$$\mu^*(E) = \mu^*(B) = \mu^*(B \cap E) + \mu^*(B \setminus E) = \mu^*(E) + \mu^*(B \setminus E).$$

Since $\mu^*(E) < \infty$, we have $\mu^*(B \setminus E) = 0$

(⇐) Suppose that there exists $B \in (A)_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0 \Rightarrow B \setminus E$ is μ^*-measurable.
Since $E = B \setminus (B \setminus E)$ and $B, B \setminus E$ are μ^*-measurable, we have E is also μ^*-measurable. ($\because \mathcal{M}(\mu^*)$ is a σ-algebra.)

(iii) Since μ is σ-finite, there exists $\{X_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n \text{ and } \mu(X_n) < \infty$$

(⇐) This proof is the same as ⇐ of (ii).

(⇒) Suppose that E is μ^*-measurable. Then $E = \bigcup_{n=1}^{\infty}(E \cap X_n)$.
Since $\mathcal{M}(\mu^*)$ is a σ-algebra and $X_n \in \mathcal{A} \subset \mathcal{M}(\mu^*)$,
Let $E_n = E \cap X_n \in \mathcal{M}(\mu^*)$, $\forall n$.
Moreover, we have $\mu^*(E_n) \le \mu^*(X_n) < \infty$, $\forall n$.
By (i), $\forall k \in \mathbb{N}$, $\exists B_{n,k} \in \mathcal{A}_\sigma$ with $B_{n,k} \supset E_n$ and

$$\mu^*(B_{n,k}) \le \mu^*(E_n) + \frac{1}{k2^n}.$$

Since $\mu^*(E_n) < \infty$ and $E_n \in M(\mu^*)$, we have

$$\mu^*(B_{n,k} \setminus E_n) = \mu^*(B_{n,k}) - \mu^*(E_n) \le \frac{1}{k2^n}$$

Therefore $\sum_{n=1}^{\infty} \mu^*(B_{n,k} \setminus E_n) \le \frac{1}{k}$.
Let $B^{(k)} = \bigcup_{n=1}^{\infty} B_{n,k} \in \mathcal{A}_\sigma \Rightarrow E \subset B^{(k)}$.

$$\mu^*(B^{(k)} \setminus E) \le \mu^*(\bigcup_{n=1}^{\infty}(B_{n,k} \setminus E_n)) \le \sum_{n=1}^{\infty} \mu^*(B_{n,k} \setminus E_n) \le \frac{1}{k}$$

Let $B = \bigcap_{k=1}^{\infty} B^{(k)}$. Then

$$\mu^*(B \setminus E) \le \mu^*(B^{(k)} \setminus E) \le \frac{1}{k}, \forall k.$$

Therefore $\mu^*(B \setminus E) = 0$ and $E \subset B$, $B \in \mathcal{A}_{\sigma\delta}$.
Hence B is what we want.

3. ($\Rightarrow$) Suppose that E is μ^*-measurable. Then by the definition of measurable sets, we have

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \setminus E)$$

Therefore, by $\mu(X) < \infty$, we obtain

$$\mu^*(E) = \mu(X) - \mu^*(E^c) = \mu_*(E)$$

($\Leftarrow$) Suppse that $\mu^*(E) = \mu_*(E)$. We have

$$\mu^*(E) = \mu_*(E) = \mu(X) - \mu^*(E^c). \tag{2}$$

We give two proofs.

proof(a): For any $F \subset X$, $\forall n \in \mathbb{N}$, $\exists A_n \in \mathcal{A}_\sigma$ with $F \subset A_n$ such that $\mu^*(A_n) \le \mu^*(F) + \frac{1}{n}$.
Thus, $\mu^*(\cap A_n) \le \mu^*(A_n) \le \mu^*(F) + \frac{1}{n}, \forall n$.
Therefore, we have $\mu^*(\cap A_n) = \mu^*(F)$.
Let $\mathcal{M}(\mu^*) =$ the collection of μ^*-measurable sets. Since $\mathcal{A} \subset \mathcal{M}(\mu^*)$ and $\mathcal{M}(\mu^*)$ is a σ-algebra, we know that $\mathcal{A}_\sigma \subset \mathcal{M}(\mu^*)$.
So, $\cap A_n \in \mathcal{M}(\mu^*)$. So far, we have the following conclusion:
For any $F \subset X$, there exsits a μ^*-measurable set$B$ such that $B \supset F$ and $\mu^*(F) = \mu^*(B)$.
Thus, we pick two measurable sets B_1, B_2 such that

$$B_1 \supset E, B_2 \supset E^c \text{ and } \mu^*(B_1) = \mu^*(E), \mu^*(B_2) = \mu^*(E^c).$$

By (2), we have $\mu(B_1) + \mu(B_2) = \mu(X)$. Since B_1 and B_2 are μ^*-measurable set, we have

$$\begin{aligned} \mu(X) = \mu(B_1) + \mu(B_2) &= \mu(B_1 \cap B_2) + \mu(B_1 \setminus B_2) + \mu(B_2) \\ &= \mu(B_1 \cap B_2) + \mu(B_1 \cup B_2) \\ &= \mu(B_1 \cap B_2) + \mu(X). \end{aligned}$$

Since $\mu(X) < \infty$, we have $\mu(B_1 \cap B_2) = 0$. Thus, $\mu(B_1 \setminus E) \le \mu(B_1 \cap B_2) = 0$. And we have $B_1 \cap B_2$ is μ^*-measurable.
By $E = B_1 \setminus (B_1 \cap B_2)$, we have E is μ^*-measurable.

proof(b): For any $F \subset X$, $\forall \varepsilon > 0$, $\exists A_\varepsilon \in \mathcal{A}_\sigma$ with $F \subset A_\varepsilon$ such that $\mu^*(A_\varepsilon) \le \mu^*(F) + \varepsilon$.
Let $\mathcal{M}(\mu^*) =$ the collection of μ^*-measurable sets. Since $\mathcal{A} \subset \mathcal{M}(\mu^*)$ and $\mathcal{M}(\mu^*)$ is a σ-algebra, we know that $\mathcal{A}_\sigma \subset \mathcal{M}(\mu^*)$.
So, $\mu(X) = \mu^*(A_\varepsilon) + \mu^*(A_\varepsilon^c)$.

By countably subadditivity of μ^*, we obtain the following inequality

$$\begin{aligned}\mu(X) &= \mu^*(A_\varepsilon) + \mu^*(A_\varepsilon^c) \\ &\leq \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c) + \mu^*(A_\varepsilon^c \cap E) + \mu^*(A_\varepsilon^c \cap E^c) \\ &= \mu^*(E) + \mu^*(E^c) \qquad \text{(because } A_\varepsilon \in M(\mu^*).) \\ &= \mu(X) \qquad \text{(by (2))}\end{aligned}$$

Therefore, we in fact have

$$\mu^*(A_\varepsilon) = \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c)$$

$$\mu^*(A_\varepsilon^c) = \mu^*(A_\varepsilon^c \cap E) + \mu^*(A_\varepsilon^c \cap E^c)$$

Since $F \cap A_\varepsilon$, we have

$$\begin{aligned}\mu^*(F \cap E) + \mu^*(F \cap E^c) &\leq \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c) \\ &= \mu^*(A_\varepsilon) \leq \mu^*(F) + \varepsilon\end{aligned}$$

Since ε is arbitrary, $\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(F)$. The opposite inequality is obvious, therefore E is μ^*-measurable.

4. (i) By the following two theorem (one is proved in the class and the other is showed in Folland,Real Analysis):

Theorem 1. *(Carathéodory extension) Let ν be a countably additive on a ring R and $\nu : R \to [0, \infty]$. There exists a measure on a σ-algebra, that coincides with ν on R. (Indeed,)*

Theorem 2. *(Folland, Real Analysis, Theorem 1.14)*

We know we can extend μ from a ring or an algebra to a σ-algebra if μ is countably additivite. But notice that $\mathcal{E}$ is not a ring! For instance, we may define $A_1 = (-2, -1] \cup (1, 2]$ and $A_2 = (-4, -3] \cup (3, 4]$. Thus it's easy to see $A_1 \cup A_2$ doesn't belog to $\mathcal{E}$.
Therefore we need to find a way to prove this problem. Here are 2 methods to prove it, but the ideas are essentially the same. Since if we write down them all, the proof becomes too long, we only show the sketches.

1) Define $\mathcal{R}$ = the ring generated by $\mathcal{E}$. Show that

$$\mathcal{R} = \{\emptyset\} \bigcup \{E | E = \bigcup_{n=1}^{m} A_{a_n,b_n}, \text{ for some m} \in \mathbb{N} \text{ and } A_{a_n,b_n} \in \mathcal{E} \text{ are mutually disjoint}\}.$$

Hence we can define $\tilde{\mu}$ on $\mathcal{R}$ by

$$\tilde{\mu}(\bigcup_{n=1}^{m} A_{a_n,b_n}) = \sum_{n=1}^{m} \mu(A_{a_n,b_n}) \text{ ,and } \tilde{\mu}(\emptyset) = 0$$

Check $\tilde{\mu}$ is countable additive on $\mathcal{R}$ and $\tilde{\mu} = \mu$ on $\mathcal{E}$.
By Theorem1, there exists a measure on a σ-algebra, that coincides with μ on $\mathcal{R}$ and therefore on $\mathcal{E}$.

2) Show that $\mathcal{E}' = \mathcal{E} \bigcup \emptyset$ is a semi-ring.
That is $\mathcal{E}'$ satisfies the following properties:
a. $\emptyset \in \mathcal{E}'$
b. $A, B \in \mathcal{E}' \Rightarrow A \cap B \in \mathcal{E}'$
c. $A, B \in \mathcal{E}' \Rightarrow \exists n \geq 0, \exists A_i \in \mathcal{E}'$ are disjoint s.t. $A \setminus B = \bigcup_{i=1}^{n} A_i$
Use the following theorem, then we can a extension of μ to a σ-algrbra:

Theorem 3. *Let $\mathcal{S}$ be a semi-ring on X and $\mu : \mathcal{S} \to [0, \infty]$ be a measure on $\mathcal{S}$. There exists a measure $\bar{\mu} : \sigma(\mathcal{S}) \to [0, \infty]$ such that $\bar{\mu} = \mu$ on $\mathcal{S}$.*

(ii) $[1, 2]$ is NOT a μ^*-measurable!
By definition of μ^*:

$$\mu^*(E) = inf\{\sum_n \mu(A)_n | E \subset \bigcup_n A_n, A_n \in \mathcal{A}\},$$

therefore $\mu^*([1, 2]) = \mu^*[-2, -1] = 1$.
Suppose to the contrary that $[1, 2]$ is μ^*-measurable, then

$$1 = \mu^*([-2, -1] \cup [1, 2]) = \mu^*([1, 2]) + \mu^*([-2, -1]) = 1 + 1 = 2$$

Therefore we get a contradiction! Hence $[1, 2]$ is not μ^*-measurable.