Real Analysis Homework #9

Due 12/8

1. Let ν be the Lebesgue measure in $[0,1]$, and let μ be the counting measure on the same σ-algebra of the Lebesgue measurable subsets of $[0,1]$. Show that ν is absolutely continuous with respect to μ. Does there exist a nonnegative μ-measurable function $f : [0,1] \to [0,\infty]$ for which $\nu(E) = \int_E f d\mu$ for any measurable set E?

2. Let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{B}_j)$, $j = 1,2$. Assume that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

3. Show that the Jordan decomposition is minimal in the sense that if μ is a signed measure and $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are measures, then $|\mu| \le \mu_1 + \mu_2$ with equality only if $\mu_1 = \mu^+$ and $\mu_2 = \mu^-$.

4. Let λ and μ be two measures on $(X, \mathcal{B})$. Suppose that μ is σ-finite and $g \ge 0$ measurable. Show that

$$g = \frac{d\lambda}{d\mu}, \quad \text{i.e.,} \quad \lambda(E) = \int_E g d\mu$$

if and only if for all $A \in \mathcal{B}$ and $\alpha, \beta \ge 0$,

$$\lambda(A \cap \{x : g(x) \ge \alpha\}) \ge \alpha\mu(A \cap \{x : g(x) \ge \alpha\}),$$

$$\lambda(A \cap \{x : g(x) < \beta\}) \le \beta\mu(A \cap \{x : g(x) < \beta\}).$$

Hint: For the "if" part, using these two conditions to show that for $\mu(A) < \infty$

$$\beta\mu(A \cap \{x : g(x) \in [\alpha, \beta)\}) \ge \lambda(A \cap \{x : g(x) \in [\alpha, \beta)\})$$

$$\ge \alpha\mu(A \cap \{x : g(x) \in [\alpha, \beta)\}).$$