Real Analysis Homework #8

Due 11/24

1. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and f a nonnegative measurable function on X. Prove that for $\lambda :=$ Lebesgue measure, $\int f d\mu = (\mu \times \lambda)\{(x, y) : 0 < y < f(x)\}$ ("the integral is the area under the curve").

2. Let $(X, \mathcal{B}, \mu)$ be σ-finite and f any measurable real-valued function on X. Prove that $(\mu \times \lambda)\{(x, y) : y = f(x)\} = 0$ (the graph of a real measurable function has measure zero).

3. Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Assume that $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing and $\phi(0) = 0$. show that

$$\int_\Omega \phi(f(x))d\mu = \int_0^\infty \phi'(t)\mu(\{x : f(x) > t\})dt.$$

4. For $I := [0, 1]$ with Borel σ-algebra and Lebesgue measure λ, take the cube I^3 with product measure (volume) $\lambda^3 = \lambda \times \lambda \times \lambda$. Let $f(x, y, z) := 1/\sqrt{|y - z|}$ for $y \neq z$, $f(x, y, z) := +\infty$ for $y = z$. Show that f is integrable for λ^3, but that for each $z \in I$, the set of y such that $\int f(x, y, z)d\lambda(x) = +\infty$ is non-empty and depends on z.