Real Analysis Homework #7

Due 11/17

1. Do the exercise given in class.

2. Let $g(x) := 1/(x \log x)$ for $x > 1$. Let $f_n = c_n 1_{A(n)}$ for some constants $c_n \geq 0$ and measurable subsets $A(n)$ of $[2, \infty)$. Prove or disprove: If $f_n(x) \to 0$ and $|f_n(x)| \leq g(x)$ for all x, then $\int_2^\infty f_n(x)dx \to 0$ as $n \to \infty$.

3. Let $f(x, y)$ be a measurable function of two real variables having a partial derivative $\partial f/\partial x$ which is bounded for $a < x < b$ and $c \leq y \leq d$, where c and d are finite and such that $\int_c^d |f(x, y)|dy < \infty$ for some $x \in (a, b)$. Prove that the integral is finite for all $x \in (a, b)$ and that we can "differentiate under the integral sign," that is, $(d/dx) \int_c^d f(x, y)dy = \int_c^d \partial f(x, y)/\partial x dy$ for $a < x < b$.

4. (a) Show that $\int_0^\infty \sin(e^x)/(1 + nx^2)dx \to 0$ as $n \to \infty$.
(b) Show that $\int_0^1 (n \cos x)/(1 + n^2 x^{3/2})dx \to 0$ as $n \to \infty$.

5. Show that if $\mu(X) < \infty$, $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n g_n \to fg$ in measure. Does the statement still hold if $\mu(X) < \infty$ is removed?

6. If $m \geq 0$ is an integer, let $J_m(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+m)!}(x/2)^{m+2n}$, Bessel function of order m.
(i) Show that if a is a constant, $2 \int_0^\infty J_m(2ax)x^{m+1}e^{-x^2}dx = a^m e^{-a^2}$.
(ii) Show that if $a > 1$, $\int_0^\infty J_0(x)e^{-ax}dx = (1 + a^2)^{-1/2}$.