Real Analysis Homework #5

Due 10/27

1. If $(X, \mathcal{S}, \mu)$ is a measure space and f a nonnegative measurable function on X, let $(f\mu)(A) := \int_A f d\mu$ for any set $A \in \mathcal{S}$.
(a) Show that $f\mu$ is a measure.
(b) If T is measurable and $1-1$ from X onto Y for a measurable space $(Y, \mathcal{A})$, with a measurable inverse T^{-1}, show that $(f\mu) \circ T^{-1} = (f \circ T^{-1})(\mu \circ T^{-1})$.

2. Let f be a simple function on $\mathbb{R}^2$ defined by $f := \sum_{j=1}^{n} j \mathbf{1}_{(j,j+2] \times (j,j+2]}$. Find the atoms of the algebra generated by the rectangles $(j, j+2] \times (j, j+2]$ for $j = 1, \cdots, n$ and express f as a sum of constants times indicator functions of such atoms.

3. Let $x \in [0, 1]$ have the expansion to the base m for some integer m, i.e., $x = 0.x_1x_2\cdots$. The non-terminating expansion is used in case of ambiguity. Show that $f_n(x) = x_n$ is a measurable function of x for each n.

4. Let $(X, \mathcal{S})$ be a measurable space and f_n any sequence of measurable functions from X into $[-\infty, \infty]$. Show that
(a) $f(x) := \sup_n f_n(x)$ defines a measurable function.
(b) $g(x) := \limsup_{n\to\infty} f_n(x)$ defines a measurable function g, as does $\liminf_{n\to\infty} f_n(x)$.

5. Prove or disprove: $|f|$ is measurable $\Rightarrow$ f is measurable.