Real Analysis Homework #2

Due 10/6

1. Let X be an infinite set. Let $\mathcal{A}$ be the collection of subsets A of X such that either A is finite, and then set $m(A) = 0$, or the complement of A is finite, and then set $m(A) = 1$.
(a) Show that $\mathcal{A}$ is an algebra but not σ-algebra.
(b) Show that m is finitely additive on $\mathcal{A}$.
(c) Under what condition on X can m be extended to a countably additive measure on a σ-algebra.

2. Given a measure space $(X, \mathcal{S}, \mu)$, let A_j and B_j be subsets of X such that $\mu^*(A_j \triangle B_j) = 0$ for all $j = 1, 2, \cdots$. Show that $\mu^*(\bigcup_j A_j) = \mu^*(\bigcup B_j)$.

3. Let (X, d) be a metric space. An outer measure μ^* on X is called a metric outer measure if

$$\rho(A_1, A_2) = \inf\{d(x_1, x_2) : x_j \in A_j\} > 0 \Rightarrow \mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Show that if μ^* is a metric outer measure, then every closed subset of X is μ^*-measurable.

4. Let $\mathcal{A}$ be an algebra and ν nonnegative, countably additive, and σ-finite on $\mathcal{A}$. Denote ν^* the outer measure associated with $(\mathcal{A}, \nu)$. Let $(X, \sigma(\mathcal{A}), \mu)$ be a measure space extended from $(\mathcal{A}, \nu)$. Denote μ^* the outer measure associated with $(\sigma(\mathcal{A}), \mu)$. Show that $\nu^* = \mu^*$.