Real Analysis Homework #10

Due 12/15

1. Let $f \in L^p(\mathbb{R}^n, dx)$ with $p > 1$. Show that

$$\lim_{r \to 0} \frac{1}{\lambda(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|^p dy = 0 \quad \text{for a.e.} \quad x,$$

where λ is the Lebesgue measure. (You can use the denseness of continuous functions with compact supports in L^p.)

2. This is an exercise on *complex measures*. A set function ν is called a complex measure if $\nu : \mathcal{B} \to \mathbb{C}$ satisfies $\nu(\emptyset) = 0$ and for each countably disjoint union $\cup E_j$, we have $\nu(\cup E_j) = \sum \nu(E_j)$ with absolute convergence on the right. Note that the infinite value is not allowed for complex measures.
(i) Show that each complex measure ν may be expressed as $\nu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_1, \cdots, \mu_4$ are finite measures.
(ii) Show that for each complex measure ν there is a measure μ and a complex-valued measurable function ψ with $|\psi| = 1$ such that for each set $E \in \mathcal{B}$,

$$\nu(E) = \int_E \psi d\mu.$$

(iii) Show that the measure μ in (ii) is unique and that ψ is uniquely determined to within sets of μ measure zero.
(iv) The measure μ in (ii) is called the total variation of absolute value of ν and is denoted by $|\nu|$. Show that if $|\nu|(X) = 1$ and $\nu(X) = 1$, then ν is a positive real measure.

3. Let E be a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{\lambda(E \cap B(r,x))}{\lambda(B(r,x))}$$

provided the limit exists.
(i) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
(ii) Find an example of E and x such that $0 < D_E(x) < 1$.

4. Let $f \in \mathcal{L}^1_{loc}$ and be continuous at x, then x is in the Lebesgue set of f.

5. If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.