Real Analysis Homework #1

Due 9/29

1. Let $\mathcal{S}$ be the σ-algebra generated by $\mathcal{A}$, then $\mathcal{S}$ is the union of the σ-algebras generated by $\mathcal{E}$ as $\mathcal{E}$ ranges over all countable subsets of $\mathcal{A}$.

2. Let $\mathcal{A} \subset 2^X$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let $\mu : \mathcal{A} \to [0, \infty]$ be countably additive on $\mathcal{A}$ and μ^* the induced outer measure.
(i) For any set $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ such that $\mu^*(A) \leq \mu^*(E) + \epsilon$.
(ii) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
(iii) Generalize (ii) to the case where μ is σ-finite.

3. As in last problem, we further assume $\mu(X) < \infty$. Let $E \subset X$, we define the inner measure of E to be $\mu_*(E) = \mu(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

4. Let $\mathcal{E}$ be the collection of the sets $A_{a,b} = [-b, -a) \cup (a, b]$ for all $0 < a < b < \infty$ and $\mathcal{A}$ the algebra generated by $\mathcal{E}$. Define $\mu(A_{a,b}) = b - a$. Show that μ extends to a countably additive measure on a σ-algebra. Is $[1, 2]$ measurable for μ^*?