CONSISTENCY OF THE BAYES METHOD FOR THE INVERSE SCATTERING PROBLEM WITH RANDOMLY TRUNCATED GAUSSIAN PRIORS

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ABSTRACT. In this work, we consider the inverse scattering problem of determining an unknown refractive index from the far-field measurements using the nonparametric Bayesian approach. This paper is a continuation of our previous work [FKW24] in which we consider Gaussian priors and Gaussian sieve priors. In this work, we will extend the result to randomly truncated Gaussian sieve priors. Our aim is to establish the consistency of the posterior distribution with an explicit contraction rate in terms of the sample size.

CONTENTS

1. INTRODUCTION

In this work, we apply the Bayes approach to study the inverse medium scattering problem. The main purpose is to prove the consistency property of the posterior distribution. This paper is a continuation of our previous work [FKW24] in which we consider Gaussian priors and Gaussian sieve priors. Here we use the same setup and notations in [FKW24]. Let $n \geq 0$ and $1 - n$ be a compactly supported function in $\mathbb{R}^3$ with $\operatorname{supp}(1-n) \subset D$, where D is an open bounded smooth domain, and having suitable regularity, which will be specified later. Let $u_n = u^{\rm inc} + u_n^{\rm sca}$ satisfy

$$\Delta u_n + k^2 n u_n = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{1.1}$$

and

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial u_n^{\rm sca}}{\partial |x|} - \mathbf{i} k u_n^{\rm sca} \right) = 0. \tag{1.2}$$

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Assume that u^{inc} is the plane incident field, i.e. $u^{\mathrm{inc}} = e^{ik\cdot\theta}$ with $\theta \in \mathbb{S}^2$. Then the scattered field u_n^{sca} possesses the asymptotic behavior

$$u_n^{\mathrm{sca}}(x,\theta) = \frac{e^{\mathbf{i}k|x|}}{|x|} u_n^{\infty}(\theta',\theta) + o(r^{-1}) \quad \text{as} \quad |x| \to \infty, \tag{1.3}$$

where $\theta' = x/|x|$. The inverse scattering problem is to determine the medium perturbation $1-n$ from the knowledge of the *scattering amplitude* $u_n^{\infty}(\theta',\theta)$ for all $\theta',\theta \in \mathbb{S}^2$ at one fixed energy k^2. It was known that the scattering amplitude $u_n^{\infty}(\theta',\theta)$ uniquely determines the refractive index n. We will not review the detailed development of the theoretical results for this problem. We only mention a log-type stability estimate derived in [HH01] which is used in this paper.

We consider the Bayesian approach to the inverse medium scattering problem here. The following measurement model was given in [FKW24]. Let μ be the uniform distribution on $\mathbb{S}^2 \times \mathbb{S}^2$, i.e., $\mu = \mathrm{d}\omega/|\mathbb{S}^2|^2$, where $\mathrm{d}\omega$ is the product measure on $\mathbb{S}^2 \times \mathbb{S}^2$, that is, $\int_{\mathbb{S}^2\times\mathbb{S}^2} \mathrm{d}\omega = |\mathbb{S}^2|^2$. We also write $\mu = \mathrm{d}\xi$ and hence $\int_{\mathbb{S}^2\times\mathbb{S}^2} \mathrm{d}\xi = 1$. Consider the iid random variables $X_i \sim \mu$, $i = 1, 2, \cdots, N$ with $N \in \mathbb{N}$.

Denote the forward map by

$$G(n)(X_i) = \begin{pmatrix} \mathrm{Re}\ (u_n^{\infty}(\theta',\theta)) \\ \mathrm{Im}\ (u_n^{\infty}(\theta',\theta)) \end{pmatrix}, \tag{1.4}$$

where (θ',θ) is a realization of X_i. The observation of the scattering amplitude $G(n)(X_i)$ is polluted by the measurement noise which is assumed to be a Gaussian random variable. Consequently, the statistical model of the scattering problem is given as

$$Y_i = G(n)(X_i) + \sigma W_i, \ \ W_i \overset{iid}{\sim} N(0, I_2), \ \ i = 1, \cdots, N, \tag{1.5}$$

where $\sigma > 0$ is the noise level, I_2 is the 2×2 unit matrix. We also assume that $W^{(N)} := \{W_i\}_{i=1}^N$ and $X^{(N)} := \{X_i\}_{i=1}^N$ are independent.

The aim here is to consider the inference of n from the observational data $(Y^{(N)}, X^{(N)})$ with $Y^{(N)} = \{Y_i\}_{i=1}^N$ using the Bayes method. We are interested in the asymptotic behavior of the posterior distribution induced from *randomly truncated Gaussian sieve priors* on n as $N \to \infty$. We would like to establish the statistical consistency theory of recovering n in (1.3) with an explicit convergence rate as the number of measurements N increases, i.e. the contraction rate of the posterior distribution to the "ground truth" n_0 when the observation data is indeed generated by n_0.

We extend the consistency results proved in [FKW24] where Gaussian process priors and Gaussian sieve priors are treated to randomly truncated Gaussian sieve priors. The setting of the problem considered in this paper is closely related to the ones studied in [GN20] and [Kek22]. In [GN20], Gaussian process priors were used in the Bayesian approach to study the recovery of the diffusion coefficient in the elliptic equation by measuring the solution at randomly chosen interior points with uniform distribution. It was shown that the posterior distribution concentrates around the true parameter at a rate $N^{-\lambda}$ for some $\lambda > 0$ as $N \to \infty$, where N is the number of measurements (or sample size). Based on the method in [GN20], similar results were proved in [Kek22] for the parabolic equation where the aim is to recover the absorption coefficient by the interior measurements of the solution.

The case of randomly truncated Gaussian sieve priors was already considered in [GN20]. However, we want to point out that we cannot directly apply the argument in there to our problem in this paper. This is due to the fact that the stability estimate for the inverse medium scattering problem is of logarithmic type. Moreover, we have to use a different link function from the one used in [GN20]. A major modification is needed in order to establish the consistency theorem with randomly truncated Gaussian sieve priors.

Both the inverse problem considered here and the one in [GN20] are reduced to the same statistical model (2.3b) through suitable link functions. Using randomly truncated sieve priors, (2.3b) gives rise to an estimate showing that the set of the unknown F with a polynomially increasing bound has a large posterior probability, see [GN20, (21) in Theorem 8]. Taking advantage of the Lipschitz stability estimate for the inverse problem considered in [GN20], such polynomial bound can be improved to a fixed constant bound [GN20, Lemma 12], which in turn leads to a polynomial contraction rate. Unfortunately, for the inverse scattering problem here, the improvement from a polynomial bound to a fixed constant bound cannot be proved due to the log-type stability estimate. We therefore take a step back to show a logarithmic contraction rate "conditioned" on a logarithmically increasing large set. To achieve this, we need to refine the stability estimate obtained in [HH01]. The refinement is to derive the explicit dependence of the constant in the stability estimate on the norm of the unknown refractive index.

This paper is organized as follows. In Section 2, we will describe the general statistical model which can be applied to the inverse medium scattering problem. In Section 3, we state the main results and their proofs are given in Section 4. In Appendix A, we present some theoretical results of the inverse scattering problem. Finally, we construct an example of the link function in Appendix B.

2. The statistical model

In order to make the paper self-contained, here we recall some notations and function spaces which used in our previous work [FKW24]. Throughout this paper, we shall use the symbol $\lesssim$ and $\gtrsim$ for inequalities holding up to a universal constant. For two real sequences (a_N) and b_N, we say that $\simeq$ if both $a_N \lesssim b_N$ and $b_N \lesssim a_N$ for all sufficiently large N. For a sequence of random variables Z_N and a real sequence (a_N), we write $Z_N = O_{\mathrm{Pr}}(a_N)$ if for all $\varepsilon > 0$ there exists $M_\varepsilon < \infty$ such that for all N large enough, $\mathrm{Pr}(|Z_N| \geq M_\varepsilon a_N) < \varepsilon$. Denote $\mathcal{L}(Z)$ the law of a random variable Z. Let $C_c^t(\mathcal{O})$ with integer $t \geq 0$ denote the Hölder space of order t with compact supports in the bounded smooth domain $\mathcal{O}$.

Let D be a bounded smooth domain in $\mathbb{R}^3$. For each integer $s \geq 0$, we denote $H^s(D)$ the standard $L^2(D)$-based Hilbert spaces, and we extend for real $s \geq 0$ by using interpolation [LM72]. It is known that the restriction operator to D is a continuous linear map of $H^s(\mathbb{R}^3)$ to $H^s(D)$ [LM72, (8.6)]. We denote $H_0^s(D)$ the completion of $C_c^\infty(D)$ in $H^s(D)$. For each compact subset $\mathcal{K} \subset \mathbb{R}^3$, we denote $H_{\mathcal{K}}^s = \{f \in H^s(\mathbb{R}^3) : \mathrm{supp}(f) \subseteq \mathcal{K}\}$. For $s > 1/2$ with $s \neq \mathbb{Z} + 1/2$, the zero extension of $f \in H_0^s(D)$ (extension of f by 0 outside of D) is a continuous map $H_0^s(D) \to H^s(\mathbb{R}^3)$ [LM72, Theorem 11.4]. In addition, $H_{\bar{D}}^s \subset H_0^s(D)$ for all $s \geq 0$ and equality (up to equivalent norms) holds when $s \notin \mathbb{Z} + 1/2$ [McL00, Theorem 3.29 and Theorem 3.33].

We now introduce the space of parameters. For integer $s \geq 0$ and $M_0 > 1$, let

$$\mathcal{F}_{M_0} = \left\{ n \in H^s(D) : 0 < n < M_0,\, n|_{\partial D} = 1,\, \frac{\partial^j n}{\partial \nu^j}\bigg|_{\partial D} = 0,\, 1 \leq j \leq s-1 \right\}. \tag{2.1}$$

For $n \in \mathcal{F}_{M_0}$, we extend $n \equiv 1$ in $\mathbb{R}^3 \setminus D$, stilled denoted by n. Then it is clear that $\mathrm{supp}\,(1-n) \subset D$. Note that for $n \in \mathcal{F}_{M_0}$, we only put the restriction on the size of n, but not on the $H^s(D)$-norm of u. As in [NvdGW20, FKW24, GN20, AN19, Kek22], we will consider a re-parametrization of $\mathcal{F}_{M_0}$ by using an appropriate link function.

Definition 2.1. Let Φ (*link function*) satisfy

(i) $\Phi : (-\infty,\infty) \to (0, M_0)$, $\Phi(0) = 1$, $\Phi'(z) > 0$ for all z;

(ii) for any $k \in \mathbb{N}$

$$\sup_{-\infty<z<\infty} |\Phi^{(k)}(z)| < \infty.$$

(iii) There exists $a > 1$ such that $\Phi'(t) \gtrsim |t|^{-a}$ when $|t|$ is sufficiently large.

The existence of a link function satisfying the requirements above is given in Appendix B. The condition $a > 1$ in (iii) is necessary, since from the restriction $0 < \Phi < M_0$, we must have

$$M_0 \geq \lim_{s\to+\infty} \Phi(s) - \Phi(t) = \lim_{s\to+\infty} \int_t^s \Phi'(\tau)\,\mathrm{d}\tau \gtrsim \lim_{s\to+\infty} \int_t^s \tau^{-a}\,\mathrm{d}\tau \quad \text{for all large } t > 1.$$

Given any link function Φ, by following from the same argument in [NvdGW20], the parameter space can be realized as (this only requires assumption (i) and (ii))

$$\mathcal{F}_{M_0} := \{\Phi(F) : F \in H_0^s(D)\}. \tag{2.2}$$

We define the reparametrized forward map by

$$\mathcal{G}(F) = G(\Phi(F)) \tag{2.3a}$$

and the statistical model (1.3) is actually a special case of the following general model:

$$Y_i = \mathcal{G}(F)(X_i) + \sigma W_i, \quad W_i \overset{iid}{\sim} N(0, I_2), \quad i = 1, \cdots, N. \tag{2.3b}$$

Assume that $\mathcal{G}$ satisfies

$$\sup_{F\in L^2(D)} \|\mathcal{G}(F)\|_{L^\infty(\mathbb{S}^2\times\mathbb{S}^2)} = S_1 < \infty, \tag{2.3c}$$

and

$$\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} \leq S_2\|F_1 - F_2\|_{L^2(D)} \quad \text{for all } F_1, F_2 \in L^2(D) \tag{2.3d}$$

for some constant $S_2 > 0$.

Remark 2.2. The assumptions (2.3c)–(2.3d) are different to those in [FKW24]. If we choose the forward map (1.4), from (A.8), we see that (2.3c) is satisfies with $S_1 = S_1(D, k, M_0)$. On the other hand, from (A.9), we have

$$\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} \leq C_3\|\Phi(F_1) - \Phi(F_2)\|_{L^2(D)}. \tag{2.4}$$

The link function Φ is (global) Lipschitz continuous (Assumption (ii) of the link function), which yields that

$$|n_1(x) - n_2(x)| = |\Phi(F_1(x)) - \Phi(F_2(x))| \lesssim |F_1(x) - F_2(x)|. \tag{2.5}$$

Combining (2.4) and (2.5), we see that (2.3d) satisfies with $S_2 = S_2(D, k, M_0)$.

Let $\mathbb{P}_F^i$ be the laws of the iid random vectors (Y_i, X_i), with expectation $\mathbb{E}_F^i$. It turns out the Radon-Nikodym derivative of $\mathbb{P}_F^i$ is given by

$$p_F(y,\xi) = \frac{\mathrm{d}\mathbb{P}_F^i}{\mathrm{d}y\times\mathrm{d}\xi} = \frac{1}{2\pi\sigma^2} e^{-\frac{(y-\mathcal{G}(F)(\xi))^T(y-\mathcal{G}(F)(\xi))}{2\sigma^2}}. \tag{2.6}$$

By slightly abusing the notation, we define $\mathbb{P}_F^N = \otimes_{i=1}^N \mathbb{P}_F^i$ the joint law of the random vectors $(Y_i, X_i)_{i=1}^N$, with expectation $\mathbb{E}_F^N$.

In the Bayesian approach, let Π be a Borel probability measure on the parameter space H_0^s supported in the Banach space $C(D)$. From the continuity property of $(F,(y,\xi)) \to p_F(y,\xi)$, the posterior distribution $\Pi(\cdot|Y^{(N)}, X^{(N)})$ of $F|(Y^{(N)}, X^{(N)})$ is given by

$$\Pi(B|Y^{(N)}, X^{(N)}) = \frac{\int_B e^{\ell^{(N)}(F)}\, \mathrm{d}\Pi(F)}{\int_{C(D)} e^{\ell^{(N)}(F)}\, \mathrm{d}\Pi(F)} \tag{2.7}$$

for any Borel set $B \subseteq C(D)$, where the log-likelihood function is written as

$$\ell^{(N)}(F) = -\frac{1}{2\sigma^2}\sum_{i=1}^N (Y_i - \mathcal{G}(F)(X_i))^T (Y_i - \mathcal{G}(F)(X_i)). \tag{2.8}$$

3. Main results

In this work we are interested in the frequentist property of the posterior distribution (2.7) in the sense that the observation data $(Y^{(N)}, X^{(N)})$ are generated through the model (1.3) of law $\mathbb{P}_{n_0}^N$ with a ground truth n_0. The aim here is to show that the posterior distribution arising from randomly truncated Gaussian sieve priors concentrates near sufficiently regular ground truth n_0 and to derive a bound on the rate of contraction. A similar result for the inverse scattering problem with Gaussian process priors and high-dimensional Gaussian sieve priors was established in [FKW24].

From computational perspective, it is useful to consider *sieve* priors that are finite-dimensional approximations of the function space supporting the prior. Here we will use a randomly truncated Karhunen-Loéve type expansion in terms of Daubechies wavelets considered in [GN20, Appendix B] or [GN21, Chapter 4]. Let $\{\Psi_{\ell r} : \ell \geq -1, r \in \mathbb{Z}^3\}$ be the (3-dimensional) compactly supported Daubechies wavelets[1], which forms an orthonormal basis of $L^2(\mathbb{R}^3)$. Let $\mathcal{K}$ be a compact set in D and for each integer $\ell \geq -1$, let $\mathcal{R}_\ell = \left\{r \in \mathbb{Z}^d : \mathrm{supp}\,(\Psi_{\ell r}) \cap \mathcal{K} \neq \emptyset\right\}$. Let $\mathcal{K}'$ be another compact subset in D such that $\mathcal{K} \subsetneq \mathcal{K}'$, and let $\chi \in C_c^\infty(D)$ be a cut-off function with $\chi = 1$ on $\mathcal{K}'$. For any real $t > 3/2$, we consider the prior (which introduced in [GN20, Section 2.2.3 and Remark 26])

$$\Pi_j = \mathcal{L}(\chi F_j), \quad F_j = \sum_{\substack{-1\leq \ell \leq j \\ r \in \mathcal{R}_\ell}} 2^{-\ell t} F_{\ell r}\Psi_{\ell r}, \quad F_{\ell r} \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$

where $j \in \mathbb{N}$ is a truncation level. According to [GN21, Exercise 2.6.5], for each $j \in \mathbb{N}$, the prior Π_j above defines a centered Gaussian probability measure supported on the finite dimensional subspace

$$\mathcal{H}_j := \mathrm{span}\,\{\chi\Psi_{\ell r} : -1 \leq \ell \leq j, r \in \mathcal{R}_\ell\},$$

with RKHS norm given in [GN20, (B2)].

Assuming that the observation data $(Y^{(N)}, X^{(N)})$ are generated through the model (1.3) of law $\mathbb{P}_{n_0}^N$. The main theme of this paper is to show that the posterior distribution arising from the statistical inverse scattering model (1.5) contracts around the "ground truth" n_0 in

[1]This can be easily constructed from the 1-dimensional Daubechies wavelets as in [GN21, Theorem 4.2.10], and the scaling functions (in the sense of [GN21, Definition 4.2.1]) is interpreted as the 'first' wavelet due to the wavelet series expansion [GN21, (4.32)].

the L^2-risk when J is a *random* truncation level (rather than the *deterministic* truncation point considered in [FKW24]), independently of the random coefficients $F_{\ell r}$, satisfying

$$\begin{aligned} \Pr(J > j) &= e^{-2^{3j}\log 2^{3j}} \quad \text{for all } j \geq 1, \\ \Pr(J = j) &\gtrsim e^{-2^{3j}\log 2^{3j}} \quad \text{as } j \to \infty, \end{aligned} \tag{3.1}$$

see [GN20, Example 28] for an example. In other words, we define Π as the law of random (conditional Gaussian) sum

$$\Pi = \mathcal{L}(\chi F), \quad F = \sum_{\substack{-1 \leq \ell \leq J \\ r \in \mathcal{R}_\ell}} 2^{-\ell t} F_{\ell r} \Psi_{\ell r}, \quad F_{\ell r} \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \tag{3.2}$$

where J is a random truncation level satisfying (3.1).

In light of the link function, we define the push-forward posterior on the refractive index n by

$$\tilde{\Pi}(\cdot|Y^{(N)}, X^{(N)}) := \mathcal{L}(n) \;\text{ with }\; n = \Phi \circ F : F \sim \Pi(\cdot|Y^{(N)}, X^{(N)}).$$

We now ready to prove our first main result by using some ideas from [GN20, MNP21].

Theorem 3.1. For each $t > 3/2$, $s \geq t$, and $N \in \mathbb{N}$, let $\xi_N := N^{-s/(2s+3)} \log N$. Suppose that the random variable J satisfies (3.1) and Π is the corresponding prior defined in (3.1). Assume that $F_0 \in H^s_K(D)$ for some $s \geq t$ and the observation $(Y^{(N)}, X^{(N)})$ to be generated through model (2.3a)–(2.3d) with the choice (1.4) of law $\mathbb{P}^N_{F_0}$, where the link function Φ in (2.3a) satisfies (i)–(iii). Let $\Pi(\cdot|Y^{(N)}, X^{(N)})$ be the posterior distribution given by (2.7). Let $0 < \epsilon < \frac{t}{t+3}$. Then for any $K > 0$, there exist constants $L > 0$ (depending on $\sigma, K, t, s, k, n_0, M_0$) and $c > 0$ (depending on D, t, k, ϵ) such that

$$\tilde{\Pi}\left(n : \begin{array}{l} \|n - n_0\|_{L^2(D)} > |\log(L\xi_N)|^{-\frac{\alpha}{2}} \\ \|1 - n\|_{H^t(D)} \leq c|\log(L\xi_N)|^{\frac{\alpha}{10}} \end{array} \middle| Y^{(N)}, X^{(N)} \right) = O_{\mathbb{P}^N_{n_0}}(e^{-KN\xi_N^2}),$$

as $N \to \infty$, where $\alpha = \frac{t}{t+3} - \epsilon > 0$ and $\mathbb{P}^N_{n_0}$ is the push-forward of $\mathbb{P}^N_{F_0}$.

To obtain an estimator of the unknown coefficient n, in view of the link function Φ, it is often convenient to derive an estimator of F. We can also prove a contraction rate for the convergence for a suitable estimator of F to F_0.

Theorem 3.2. Assume that the hypotheses of Theorem 3.1 hold. There exists a constant $c' > 0$ (depending on $\Phi, D, t, s, k, \epsilon, \sigma, F_0, K$) such that the (Bochner) estimator

$$\overline{F}_N := \mathbb{E}^\Pi \left[F \mathbb{1}_{\|F\|_{H^t(D)} \leq c'^{\frac{1}{t}} |\log(L\xi_N)|^{\frac{\alpha}{10t}}} | Y^{(N)}, X^{(N)} \right]$$

satisfies

$$\mathbb{P}^N_{F_0}\left(\|\overline{F}_N - F_0 Z_N\|_{L^2(D)} > C' |\log(L\xi_N)|^{-\frac{\alpha}{2}(1 - \frac{a}{5t})} \right) \to 0 \quad \text{as } N \to \infty \tag{3.3}$$

provided the addition condition $1 < a < 5$ assumed in (iii), where

$$Z_N = \Pi\left(\|F\|_{H^t(D)} \leq c'^{\frac{1}{t}} |\log(L\xi_N)|^{\frac{\alpha}{10t}} | Y^{(N)}, X^{(N)} \right)$$

and the constant C' depends on D, a, t, k, ϵ.

Roughly speaking, $\overline{F}_N$ can be regarded as the expectation of the posterior distribution $\Pi\left(\cdot|Y^N, X^N\right)$ "conditioned" on $\|F\|_{H^t(D)} \leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}$. Denote the set $A_N := \left\{F \in H_0^t(D) : \|F\|_{H^t(D)} \leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}\right\}$ and $\tilde{A}_N$ the push forward of A_N by the link function Φ. Moreover, let $\tilde{Z}_N = \tilde{\Pi}\left(\tilde{A}_N|Y^{(N)}, X^{(N)}\right)$ be the posterior measure of $\tilde{A}$. It is easy to observe that $\mathcal{F}_{M_0} \setminus \tilde{A}_N \downarrow 0$ as $N \to \infty$.

Corollary 3.3. Under the hypotheses of Theorem 3.2, we have

$$\mathbb{P}_{n_0}^N\left(\|\Phi \circ \overline{F}_N - n_0\tilde{Z}_N\|_{L^2(D)} > C|\log(L\xi_N)|^{-\frac{\alpha}{2}(1-\frac{a}{5t})}\right) \to 0 \quad \text{as} \quad N \to \infty.$$

Corollary 3.3 shows that $\Phi\circ\overline{F}_N$ is an efficient estimator of n_0 conditioned on the sufficiently "large" set $\tilde{A}_N$.

4. Proof of Theorems

We will need the following proposition concerning the general contraction rate, without referring to the composition operator (2.3a).

Proposition 4.1 ([GN20, Theorem 19])**.** Suppose that the hypotheses of Theorem 3.1 are satisfied. Then given any $K > 0$, there exist sufficiently large constants $L > 0$ (depending on $\sigma, F_0, K, t, S_1, S_2$) such that

$$\Pi(F : \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} > L\xi_N|Y^{(N)}, X^{(N)}) = O_{\mathbb{P}_{F_0}^N}(e^{-KN\xi_N^2})$$

as $N \to \infty$.

We now prove Theorem 3.1 by considering the form of the forward map (2.3a) in Proposition 4.1.

Proof of Theorem 3.1. For each $M > 0$ satisfies $\|1-n\|_{H^t(D)} \vee \|1-n_0\|_{H^t(D)} \leq M$, one has the stability estimate of G^{-1} in Theorem A.5:

$$\|n - n_0\|_{L^2(D)} \leq CM^5|\log\|G(n) - G(n_0)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}|^{-\alpha}. \tag{4.1}$$

where $\alpha = \frac{t}{t+3} - \epsilon > 0$ and $C = C(D, M_0, t, k, \epsilon)$, both independent of the parameter M. In view of (4.1) and Proposition 4.1, given any $K > 0$, there exist a large constant L (which is independent of M) such that

$$\begin{aligned}
&\tilde{\Pi}\left(n : \|n - n_0\|_{L^2(D)} > |\log(L\xi_N)|^{-\frac{\alpha}{2}}, \|1-n\|_{H^t(D)} \leq M|Y^{(N)}, X^{(N)}\right) \\
&\quad\leq \tilde{\Pi}\left(n : |\log(L\xi_N)|^{-\frac{\alpha}{2}} < CM^5|\log\|G(n) - G(n_0)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}|^{-\alpha}|Y^{(N)}, X^{(N)}\right) \\
&\quad= \Pi\left(F : |\log(L\xi_N)|^{-\frac{\alpha}{2}} < CM^5|\log\|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}|^{-\alpha}|Y^{(N)}, X^{(N)}\right).
\end{aligned}$$

We now set $M > 0$ satisfying $CM^5 = |\log(L\xi_N)|^{\frac{\alpha}{2}}$. From the inequality above, it yields

$$\begin{aligned}
&\tilde{\Pi}\left(n : \begin{array}{l}\|n - n_0\|_{L^2(D)} > |\log(L\xi_N)|^{-\frac{\alpha}{2}} \\ \|1-n\|_{H^t(D)} \leq c|\log(L\xi_N)|^{\frac{\alpha}{10}}\end{array}\middle| Y^{(N)}, X^{(N)}\right) \\
&\quad\leq \Pi(F : \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} > L\xi_N|Y^{(N)}, X^{(N)}) = O_{\mathbb{P}_{F_0}^N}(e^{-KN\xi_N^2}),
\end{aligned} \tag{4.2}$$

which concludes the theorem with $c = C^{-5}$. Note that $\xi_N \to 0$ as $N \to \infty$. ∎

Having proved Theorem 3.1, we then establish Theorem 3.2 using the contraction rate in Theorem 3.1 and the link function Φ, by following the same ideas as in [GN20, Theorem 6]. However, here we need to keep track of the dependence of $\|F\|_{H^t(D)}$ carefully.

Proof of Theorem 3.2. Let $c' > 0$ be a number to be determined later and we aim to estimate $\mathbb{E}^{\Pi}\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right]$. Since $n = \Phi\circ F$ and $n_0 = \Phi\circ F_0$, by the property (i) of Definition 2.1, the mean-value theorem and inverse function theorems, there exists η lying between $n_0(x)$ and $n(x)$ such that

$$|F(x)-F_0(x)| = \frac{1}{|\Phi'(\Phi^{-1}(\eta))|}|n(x)-n_0(x)| \quad \text{for all } x\in D.$$

If $\|F\|_{L^\infty(D)} \lesssim c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}$ and $\|F_0\|_{L^\infty(D)} \lesssim c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}$, by (i) and (iii) of Definition 2.1, we have

$$\begin{aligned}|F(x)-F_0(x)| &\leq \frac{1}{\min_{z\in[-c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}},c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}]}\Phi'(z)}|n(x)-n_0(x)| \\ &\lesssim c'^{\frac{a}{t}}|\log(L\xi_N)|^{\frac{a\alpha}{10t}}|n(x)-n_0(x)|.\end{aligned}\tag{4.3}$$

In deriving (4.3), we note that $\Phi'(z)>0$ for all z and the minimum of $\Phi'(z)$ on any compact interval $[-r,r]$ decays at the rate of $|r|^{-a}$ (see (iii) of Definition 2.1). On the other hand, it follows from [NvdGW20, Lemma 29] that

$$\|1-n\|_{H^t(D)} = \|\Phi(0)-\Phi(F)\|_{H^t(D)} \leq C_0(1+\|F\|_{H^t(D)}^t),$$

where $C_0 = C_0(D,\Phi,t)$. For N large, we can see that if $\|F\|_{H^t(D)}\leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}$, then

$$\|1-n\|_{H^t(D)} \leq 2C_0c'|\log(L\xi_N)|^{\frac{\alpha}{10}}.$$

Therefore, by choosing $c' = (2C_0)^{-1}c$, where c is the constant derived in Theorem 3.1, we obtain

$$\begin{aligned}&\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}} \\ &\quad\leq C|\log(L\xi_N)|^{\frac{a\alpha}{10t}}\|n-n_0\|_{L^2(D)}\mathbb{1}_{\|1-n\|_{H^t(D)}\leq c|\log(L\xi_N)|^{\frac{\alpha}{10}}},\end{aligned}$$

where C depends on D,a,t,k,ϵ. Consequently, we can show that

$$\begin{aligned}&C^{-1}|\log(L\xi_N)|^{-\frac{a\alpha}{10t}}\mathbb{E}^{\Pi}\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\leq c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] \\ &\leq \mathbb{E}^{\tilde{\Pi}}\left[\|n-n_0\|_{L^2(D)}\mathbb{1}_{\|1-n\|_{H^t(D)}\leq c|\log(L\xi_N)|^{\frac{\alpha}{10}}}|Y^{(N)},X^{(N)}\right] \\ &\leq |\log(L\xi_N)|^{-\frac{\alpha}{2}} \\ &\quad+\mathbb{E}^{\tilde{\Pi}}\left[\|n-n_0\|_{L^2(D)}\mathbb{1}_{\|n-n_0\|_{L^2(D)}>|\log(L\xi_N)|^{-\frac{\alpha}{2}}}\mathbb{1}_{\|1-n\|_{H^t(D)}\leq c|\log(L\xi_N)|^{\frac{\alpha}{10}}}|Y^{(N)},X^{(N)}\right] \\ &\leq |\log(L\xi_N)|^{-\frac{\alpha}{2}}+\sqrt{\mathbb{E}^{\tilde{\Pi}}\left[\|n-n_0\|_{L^2(D)}^2\middle|Y^{(N)},X^{(N)}\right]}\times \\ &\quad\times\sqrt{\tilde{\Pi}\left(n: \begin{matrix}\|n-n_0\|_{L^2(D)}>|\log(L\xi_N)|^{-\frac{\alpha}{2}} \\ \|1-n\|_{H^t(D)}\leq c|\log(L\xi_N)|^{\frac{\alpha}{10}}\end{matrix}\middle|Y^{(N)},X^{(N)}\right)}.\end{aligned}\tag{4.4}$$

Let $0 < B < K$ be any fixed constant, then

$$
\begin{aligned}
&\mathbb{P}^N_{F_0}\left(\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] > 2C|\log(L\xi_N)|^{-\frac{\alpha}{2}(1-\frac{a}{5t})}\right)\\
&=\mathbb{P}^N_{F_0}\left(C^{-1}|\log(L\xi_N)|^{-\frac{a\alpha}{10t}}\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] > 2|\log(L\xi_N)|^{-\frac{\alpha}{2}}\right)\\
&=\mathbb{P}^N_{F_0}\begin{pmatrix}C^{-1}|\log(L\xi_N)|^{-\frac{a\alpha}{10t}}\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] > 2|\log(L\xi_N)|^{-\frac{\alpha}{2}},\\
\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} > |\log(L\xi_N)|^{-\alpha}\end{pmatrix}\\
&\quad+\mathbb{P}^N_{F_0}\begin{pmatrix}C^{-1}|\log(L\xi_N)|^{-\frac{a\alpha}{10t}}\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] > 2|\log(L\xi_N)|^{-\frac{\alpha}{2}},\\
\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} \le |\log(L\xi_N)|^{-\alpha}\end{pmatrix}\\
&\le\mathbb{P}^N_{n_0}\left(\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} > |\log(L\xi_N)|^{-\alpha}\right)\\
&\quad+\mathbb{P}^N_{n_0}\begin{pmatrix}\sqrt{\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]}\\
\times\sqrt{\tilde\Pi\left(n:\begin{array}{l}\|n-n_0\|_{L^2(D)}>|\log(L\xi_N)|^{-\frac{\alpha}{2}}\\ \|1-n\|_{H^t(D)}\le c|\log(L\xi_N)|^{\frac{\alpha}{10}}\end{array}\Bigg|Y^{(N)},X^{(N)}\right)} \ge |\log(L\xi_N)|^{-\frac{\alpha}{2}}\\
\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} \le |\log(L\xi_N)|^{-\alpha}\end{pmatrix}\\
&\le\mathbb{P}^N_{n_0}\left(\mathbb{E}^{\tilde\Pi}\left[\|n-n_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} > |\log(L\xi_N)|^{-\alpha}\right)\\
&\quad+\mathbb{P}^N_{n_0}\left(\tilde\Pi\left(n:\begin{array}{l}\|n-n_0\|_{L^2(D)}>|\log(L\xi_N)|^{-\frac{\alpha}{2}}\\ \|1-n\|_{H^t(D)}\le c|\log(L\xi_N)|^{\frac{\alpha}{10}}\end{array}\Bigg|Y^{(N)},X^{(N)}\right)\ge e^{-BN\xi_N^2}\right)\\
&=\mathbb{P}^N_{F_0}\left(\mathbb{E}^{\Pi}\left[\|F-F_0\|^2_{L^2(D)}\Big|Y^{(N)},X^{(N)}\right]e^{-BN\xi_N^2} > |\log(L\xi_N)|^{-\alpha}\right)+o(1),
\end{aligned}
$$

where we have used (4.4) and then (4.2). Likewise, we can show that the first probability on the right hand side above vanishes as $N\to\infty$ by proceeding as in the proof of [GN20, Theorem 6, page 14] (there, replacing [GN20, Lemma 16] by [GN20, Lemma 20]). Note that for Π the random series prior given in (3.2), it also holds that $\mathbb{E}^\Pi\|F\|^2_{L^2(D)}<\infty$, see [GN20, page 17]. In other words, we can obtain that

$$
\mathbb{P}^N_{F_0}\left(\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right] > 2C|\log(L\xi_N)|^{-\frac{\alpha}{2}(1-\frac{a}{5t})}\right)\to 0.
$$

Finally, by the definition of $\overline{F}_N$ and Minkowski's inequality, we see that

$$
\begin{aligned}
&\|\overline{F}_N-F_0Z_N\|_{L^2(D)}\\
&=\left(\int_D\left(\mathbb{E}^\Pi\left[(F-F_0)\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right]\right)^2\mathrm{d}x\right)^{1/2}\\
&\le\mathbb{E}^\Pi\left[\|F-F_0\|_{L^2(D)}\mathbb{1}_{\|F\|_{H^t(D)}\le c'^{\frac{1}{t}}|\log(L\xi_N)|^{\frac{\alpha}{10t}}}|Y^{(N)},X^{(N)}\right],
\end{aligned}
$$

which concludes our result. □

Appendix A. Inverse scattering problem

A.1. **Forward problem.** In what follows, instead of (2.1), we only assume that

$$n \in L^2(D), \ \ \mathrm{supp}(1-n) \subset D, \ \ 0 < n \leq M_0 \ \text{ for some } \ M_0 > 0. \tag{A.1}$$

We want to show the existence and uniqueness of u_n^{sca} by refining the ideas in [FKW24, Appendix A]. It is known that $w := u_n^{\mathrm{sca}}$ satisfies the following boundary value problem:

$$\begin{cases} -\Delta w - k^2 n w = k^2(n-1)u^{\mathrm{inc}} & \text{in } \ B_R, \\ \partial w/\partial r = S_R(w|_{\partial B_R}) & \text{on } \ \partial B_R, \end{cases} \tag{A.2}$$

where B_R is a ball of radius R such that $\bar{D} \subset B_R$. Here $S_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann map, defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g = (\partial u_g/\partial r)|_{\partial B_R}$, where u_g is the solution of the Helmholtz equation satisfying the Sommerfeld radiation condition in $R^3 \setminus B_R$ and the Dirichlet condition $u_g = g$ on ∂B_R. It has been shown that

$$\mathrm{Re}\langle S_R(v), v\rangle \leq 0 \quad \text{and} \quad \mathrm{Im}\langle S_R(v), v\rangle \geq 0, \ \ \forall \ v \in H^{1/2}(\partial B_R), \tag{A.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\partial B_R)$ and $H^{1/2}(\partial B_R)$, see, for example, [CCH23, Definition 1.36 and (1.50)].

Let us replace the right-hand side of the first equation in (A.2) by a general source term f with $\mathrm{supp}(f) \subset B_R$, i.e.,

$$\begin{cases} -\Delta w - k^2 n w = f & \text{in } \ B_R, \\ \partial w/\partial r = S_R(w|_{\partial B_R}) & \text{on } \ \partial B_R, \end{cases} \tag{A.4}$$

In view of the integration by parts, (A.4) is equivalent to the following variational formulation: find $w \in H^1(B_R)$ such that for all $v \in H^1(B_R)$,

$$a_1(w,v) + a_2(w,v) = F(v),$$

where

$$a_1(w,v) = \int_{B_R} \nabla w \cdot \nabla \bar{v}\,\mathrm{d}x - k^2 \int_{B_R} n w \bar{v}\,\mathrm{d}x + k^2(M_0+1)\int_{B_R} w\bar{v}\,\mathrm{d}x - \langle S_R(w), \bar{v}\rangle,$$

$$a_2(w,v) = -k^2(M_0+1)\int_{B_R} w\bar{v}\,\mathrm{d}x,$$

and

$$F(v) = \int_{B_R} f\bar{v}\,\mathrm{d}x.$$

By using (A.3), we can see that $a_1(\cdot,\cdot)$ is strictly coercive in the sense of

$$\mathrm{Re}\, a_1(w,w) = \int_{B_R} \left(|\nabla w|^2 + k^2(M_0+1-n)|w|^2\right)\,\mathrm{d}x \geq C(k)\|w\|^2_{H^1(B_R)}.$$

We now define the bounded linear operator $\mathcal{A} : H^1(B_R) \to (H^1(B_R))^*$, where $(H^1(B_R))^*$ is the dual space of $H^1(B_R)$, by

$$\langle \mathcal{A}w, v\rangle := a_1(w,v) \quad \text{for all } v \in H^1(B_R),$$

where $\langle\cdot,\cdot\rangle$ is the $(H^1(B_R))^* \times H^1(B_R)$ duality pair. We remind the readers that

$$(H^1(B_R))^* \neq (H_0^1(B_R))^* = H^{-1}(B_R).$$

By using the Lax-Milgram theorem, one sees that the operator $\mathcal{A} : H^1(B_R) \to (H^1(B_R))^*$ is invertible. Since bounded linear operator $\mathcal{B} : H^1(B_R) \to (H^1(B_R))^*$ given by

$$\langle \mathcal{B}w, v\rangle := a_2(w,v) \quad \text{for all } v \in H^1(B_R)$$

is compact, then $\mathcal{A}+\mathcal{B}$ is a Fredholm operator. By the Fredholm alternative, $\mathcal{A}+\mathcal{B} : H^1(B_R) \to (H^1(B_R))^*$ is bounded invertible provided the kernel of $\mathcal{A}+\mathcal{B}$ is trivial, which follows the uniqueness of the scattered solution[2]. In addition, there exists a constant $C = C(D,k,M_0) > 0$ such that

$$\|w\|_{H^1(B_R)} \leq C\|f\|_{(H^1(B_R))^*} \leq C\|f\|_{L^2(B_R)}. \tag{A.5}$$

We now choose $f = k^2(n-1)u^{\text{inc}}$ and $w = u_n^{\text{sca}}$ in (A.5) to obtain

$$\sup_{\theta\in\mathbb{S}^2} \|u_n^{\text{sca}}(\cdot,\theta)\|_{H^1(B_R)} \leq C_1\|1-n\|_{L^2(D)}$$

with $C_0 = C_0(D,k,M_0) > 0$, which is independent of $\theta \in \mathbb{S}^2$. Note that $\text{supp}(f) \subset D \subset B_R$. By the interior estimate [GT01, Theorem 8.8] and the Sobolev embedding theorem, we reach

$$\sup_{\theta\in\mathbb{S}^2} \|u_n^{\text{sca}}(\cdot,\theta)\|_{C(\overline{D})} \leq C \sup_{\theta\in\mathbb{S}^2} \|u_n^{\text{sca}}(\cdot,\theta)\|_{H^2(B_R)} \leq C_0\|1-n\|_{L^2(D)} \tag{A.6}$$

with $C_1 = C_1(D,k,M_0) > 0$, which is also independent of $\theta \in \mathbb{S}^2$.

The scattering amplitude $u_n^\infty(\theta',\theta)$ can be expressed explicitly by

$$u_n^\infty(\theta',\theta) = -\frac{k^2}{4\pi}\int_D e^{-\mathbf{i}k\theta'\cdot y}(1-n)(y)u(y,\theta)\,\mathrm{d}y, \tag{A.7}$$

where $u(y,\theta) = u^{\text{inc}}(y,\theta) + u_n^{\text{sca}}(y,\theta)$ is the total field and $u^{\text{inc}}(y,\theta) = e^{\mathbf{i}ky\cdot\theta}$, see [CK19, (8.28)] or [CCH23, (1.22)] or [Ser17, Page 232]. We combine (A.6) and (A.7) to obtain

$$\|u_n^\infty\|_{L^\infty(\mathbb{S}^2\times\mathbb{S}^2)} \leq C_2(1+\|n\|_{L^2(D)}^2) \leq S \tag{A.8}$$

for some constant $C_2 = C_2(D,k,M_0)$ and $S = S(D,k,M_0)$, which implies the uniform boundedness condition (2.3c).

Let n_1, n_2 satisfy (A.12). We choose $w = u_{n_2}^{\text{sca}} - u_{n_1}^{\text{sca}}$ and $f = k^2(n_2-n_1)u_{n_1}^{\text{sca}} + k^2(n_2-n_1)u^{\text{inc}}$ in (A.5), as well as using (A.6), to obtain

$$\|u_{n_1}^{\text{sca}} - u_{n_2}^{\text{sca}}\|_{H^1(B_R)} \leq C\left(1+\|n_1\|_{L^2(D)} + \|n_2\|_{L^2(D)}\right)\|n_1-n_2\|_{L^2(D)}$$

for some constant $C = C(D,k,M_0) > 0$. From (A.7) and the equation above it is not difficult to see that

$$\|u_{n_1}^\infty - u_{n_2}^\infty\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} \leq C_3\left(1+\|n_1\|_{L^2(D)} + \|n_2\|_{L^2(D)}\right)\|n_1-n_2\|_{L^2(D)} \tag{A.9}$$

for some constant $C_3 = C_3(D,k,M_0) > 0$.

[2]The proof needs Rellich's lemma [CCH23, page 6] and the unique continuation property [CCH23, page 11], which is only required $n \in L^2(D)$.

A.2. **Inverse problem.** Next, we want to refine the stability estimate in [HH01, Theorem 1.2], which studies the determination of the potential from the knowledge of the scattering amplitude. To do so, we first refine [Häh96, Lemma 5] in the following lemma. Even though the proof is similar, here we still present the proof to make the paper self contained.

Lemma A.1. Let $d \geq 2$ and $R_0 > 0$. Assume that $B_{r_0} \subset \mathbb{R}^d$ is an open ball centered at the origin with radius $r_0 \leq R_0$. Let $q \in L^\infty(\mathbb{R}^d)$ satisfy $\mathrm{supp}\,(q) \subset \overline{B_{r_0}}$ and $k \geq 0$ be fixed. Given any $\zeta \in \mathbb{C}^d$ with $\zeta \cdot \zeta = k^2$ and $|\mathrm{Im}\,(\zeta)| \geq (R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)} + 1$, there exists a solution $u \in H^2(B_{r_0})$ of $(\Delta + k^2 + q(x))u = 0$ in B_{r_0} such that $u(x) = e^{\mathbf{i}\zeta\cdot x}(1 + w_\zeta(x))$ for all $x \in B_{r_0}$ with

$$\|w_\zeta\|_{L^2(B_{r_0})} \leq (2R_0)^{\frac{d}{2}}(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}\left((R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)} + 1\right)|\mathrm{Im}\,\zeta|^{-1}.$$

Proof. Let $\zeta \in \mathbb{C}^d$ be given in the lemma. Since $\mathrm{Im}\,\zeta \cdot \mathrm{Re}\,\zeta = 0$, one can find an orthogonal transform $Q \in \mathbb{R}^{d\times d}$ such that $Q(\mathrm{Re}\,\zeta) = (|\mathrm{Re}\,\zeta|, 0, \cdots, 0)^\intercal$ and $Q(\mathrm{Im}\,\zeta) = (0, |\mathrm{Im}\,\zeta|, 0, \cdots, 0)^\intercal$. We choose

$$\xi := (|\mathrm{Re}\,\zeta|, \mathbf{i}|\mathrm{Im}\,\zeta|, 0, \cdots, 0)^\intercal, \quad \tilde{q}(x) := q(Q^\intercal x) \text{ for } x \in \mathbb{R}^d.$$

By [Häh96, Theorem 1], given any $f \in L^2((-R_0, R_0)^d)$ one has the estimate

$$\begin{aligned}\|G_\xi(\tilde{q}f)\|_{L^2((-R_0,R_0)^d)} &\leq \frac{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}}{|\mathrm{Im}\,\zeta|}\|f\|_{L^2((-R_0,R_0)^d)} \\ &\leq \frac{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}}{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)} + 1}\|f\|_{L^2((-R_0,R_0)^d)},\end{aligned}$$

where $G_\xi : L^2((-R_0, R_0)^d) \to H^2((-R_0, R_0)^d)$ is the operator defined in [Häh96, Theorem 1]. By Banach's fixed point theorem, we can find a unique solution to $w_\xi \in L^2((-R_0, R_0)^d)$ of $w_\xi = -G_\xi(\tilde{q}w_\xi + \tilde{q})$. Furthermore, we have $w_\xi \in H^2((-R_0, R_0)^d)$ and

$$\begin{aligned}\|w_\xi\|_{L^2((-R_0,R_0)^d)} &= \|G_\xi(\tilde{q}w_\xi + \tilde{q})\|_{L^2((-R_0,R_0)^d)} \\ &\leq \frac{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}}{|\mathrm{Im}\,\zeta|}\|w_\xi + 1\|_{L^2((-R_0,R_0)^d)} \\ &\leq \frac{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}}{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)} + 1}\|w_\xi\|_{L^2((-R_0,R_0)^d)} + (2R_0)^{\frac{d}{2}}\frac{(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}}{|\mathrm{Im}\,\zeta|},\end{aligned}$$

which implies

$$\|w_\xi\|_{L^2((-R_0,R_0)^d)} \leq (2R_0)^{\frac{d}{2}}(R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)}\left((R_0/\pi)\|q\|_{L^\infty(\mathbb{R}^d)} + 1\right)|\mathrm{Im}\,\zeta|^{-1}.$$

Finally, one can verify that $u(x) := e^{\mathbf{i}\zeta\cdot x}(1 + w_\zeta(x))$ for $x \in B_{r_0}$, where $w_\zeta(x) := w_\xi(Qx)$, is our desired solution. □

Let us introduce the point source $u^{\mathrm{inc}}(x) = \Xi(x, y)$ located at $y \in \mathbb{R}^3$ with $|y| > R$ as incident fields, where

$$\Xi(x, y) := \frac{1}{4\pi}\frac{e^{\mathbf{i}k|x-y|}}{|x - y|}$$

is the outgoing fundamental solution to the Helmholtz operator $-(\Delta + k^2)$. Let $w_n^{\mathrm{sca}}(\cdot, y)$ be the solution to (A.4) with u^{inc} given above and denote the total field by $w_n(\cdot, y) =$

$\Xi(\cdot,y)+w_n^{\mathrm{sca}}(\cdot,y)$, which is the Green's function of the corresponding scattering problem. We now introduce the operator

$$(S_n\varphi)(x) := \int_{\partial B_R} w_n(x,y)\varphi(y)\,\mathrm{d}s(y), \quad x\in\partial B_R$$

see [HH01, Lemma 2.1] for more details.

For each $y\in\partial B_R$, we choose $f=k^2(n-1)\Xi(\cdot,y)$ (note $\mathrm{supp}(f)\subset D\subset B_R$) and $w=w_n^{\mathrm{sca}}(\cdot,y)$ in (A.5) to derive

$$\sup_{y\in\partial B_R}\|w_n(\cdot,y)-\Xi(\cdot,y)\|_{L^2(\partial B_R)} \le \sup_{y\in\partial B_R}\|w_n(\cdot,y)-\Phi(\cdot,y)\|_{H^1(B_R)} \le C_0$$

for some constant $C_0=C_0(D,k,M_0,R)$, where we have used the observation

$$\begin{aligned}\int_D |f|^2\,\mathrm{d}x &\le k^4\int_{B_R}|n-1|^2|\Xi(x,y)|^2\,\mathrm{d}x\\ &\le k^4(M_0+1)^2\int_{B_{2R}}|\Xi(x,y)|^2\,\mathrm{d}x \le Ck^4(M_0+1)^2\end{aligned} \tag{A.10}$$

with $C=C(R)$. This implies that

$$\|w_n-\Xi\|_{L^2(\partial B_R\times\partial B_R)} \le C_0$$

for some constant $C_0=C_0(D,k,M_0,R)$. Therefore, for n_1,n_2 satisfying (A.1), we have

$$\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)} \le C_0 \tag{A.11}$$

with $C_0=C_0(D,k,M_0,R)$.

Next we recall the following lemma from [HH01].

Lemma A.2. [HH01, Lemma 3.2] Let $D\subset B_R$. Assume that n_1,n_2 are refractive indices with $\mathrm{supp}\,(1-n_j)\subset D$ for $j=1,2$. Then there exists a positive constant $C=C(k,R)$ such that

$$\left|\int_D (n_1-n_2)u_1u_2\,\mathrm{d}x\right| \le C\|S_{n_1}-S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)}\|u_1\|_{L^2(B_{2R})}\|u_2\|_{L^2(B_{2R})}$$

for all solutions $u_j\in H^2(B_{2R})$ of $(\Delta+k^2n_j)u_j=0$ in B_{2R}.

Let $t>\frac{3}{2}$ and n_1,n_2 satisfy

$$\mathrm{supp}\,(1-n_j)\subset D \ \text{ and } \ \|1-n_j\|_{H^t}\le M, \ \ j=1,2, \tag{A.12}$$

where $M\ge M_0$ and we assume $M\ge 1$. We have

$$n_1(x)-n_2(x) = (2\pi)^{-\frac{3}{2}}\sum_{\gamma\in\mathbb{Z}^3}(n_1-n_2)\widehat{\ }(\gamma)e^{\mathbf{i}\gamma\cdot x}, \tag{A.13}$$

where $\hat{f}$ denotes the Fourier coefficients of a function $f\in L^2((-\pi,\pi)^3)$ with respect to the orthonormal bases $(2\pi)^{-\frac{3}{2}}e^{\mathbf{i}\gamma\cdot x}$ for $x\in(-\pi,\pi)^3$ and $\gamma\in\mathbb{Z}^3$. Given any constant $\rho\ge 2$, it is easy to see that

$$\sum_{|\gamma|>\rho}|(n_1-n_2)\widehat{\ }(\gamma)|^2 \le \frac{1}{(1+\rho^2)^t}\sum_{|\gamma|>\rho}(1+\gamma\cdot\gamma)^t|(n_1-n_2)\widehat{\ }(\gamma)|^2 \le \frac{CM^2}{\rho^{2t}} \tag{A.14}$$

and

$$\sum_{|\gamma|>\rho} |(n_1 - n_2)\widehat{\ }(\gamma)| \tag{A.15}$$

$$\leq \left(\sum_{|\gamma>\rho|} (1 + \gamma \cdot \gamma)^t |(n_1 - n_2)\widehat{\ }(\gamma)|^2 \right)^{\frac{1}{2}} \left(\sum_{|\gamma|>\rho} \frac{1}{(1 + \gamma \cdot \gamma)^t} \right)^{\frac{1}{2}} \leq \frac{CM}{\rho^{t-\frac{3}{2}}}.$$

for some $C = C(t) > 0$, see [HH01, Page 675]. We now estimate the Fourier coefficients $(n_1 - n_2)\widehat{\ }(\gamma)$ for $|\gamma| \leq \rho$ as in [HH01, Lemma 3.3].

Lemma A.3. Let $t > \frac{3}{2}$, $\rho \geq 2$ and n_1, n_2 satisfy (A.12). Moreover, define $\tau_0 := \sqrt{M_1^2 + k^2}$ with $M_1 = k^2 M + 1$. Then there exists a constant $C = C(k, R) > 0$ such that

$$|(n_1 - n_2)\widehat{\ }(\gamma)| \leq CM^5 \left(e^{4R(\tau+\rho)} \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R) \to L^2(\partial B_R)} + \frac{1}{\tau} \right)$$

for all $\gamma \in \mathbb{Z}^3$ with $|\gamma| \leq \rho$ and for all $\tau > \tau_0$.

Proof. We choose unit vectors $d_1, d_2 \in \mathbb{R}^3$ such that $d_1 \cdot d_2 = d_1 \cdot \gamma = d_2 \cdot \gamma = 0$ and define the complex vectors

$$\zeta_{1,\tau} := -\frac{1}{2}\gamma + \mathbf{i}\sqrt{\tau^2 - k^2 + \frac{|\gamma|^2}{4}} d_1 + \tau d_2 \in \mathbb{C}^3,$$
$$\zeta_{2,\tau} := -\frac{1}{2}\gamma - \mathbf{i}\sqrt{\tau^2 - k^2 + \frac{|\gamma|^2}{4}} d_1 - \tau d_2 \in \mathbb{C}^3.$$

Then $\zeta_{1,\tau} + \zeta_{2,\tau} = -\gamma$ and $\zeta_{1,\tau} \cdot \zeta_{1,\tau} = \zeta_{2,\tau} \cdot \zeta_{2,\tau} = k^2$. Since $\tau > \tau_0$, we have

$$|\mathrm{Im}\,(\zeta_{j,\tau})| \geq M_1 \quad \text{for all } j = 1, 2.$$

Using Lemma A.1 with $d = 3$, $q_j = k^2(n_j - 1)$, and $r_0 = R_0 = 2R$, for $j = 1, 2$, there exist geometrical optics solutions

$$u_{j,\tau}(x) = e^{\mathbf{i}\zeta_{j,\tau} \cdot x}(1 + v_{j,\tau}(x)) \text{ for } x \in B_{2R}, \quad \|v_{j,\tau}\|_{L^2(B_{2R})} \leq CM^2 |\mathrm{Im}\,\zeta_{j,\tau}|^{-1}$$

for some $C = C(k, R') > 0$. We can see that

$$u_{1,\tau}(x) u_{2,\tau}(x) = e^{-\mathbf{i}\gamma \cdot x}(1 + p_\tau(x)), \quad p_\tau(x) = v_{1,\tau}(x) + v_{2,\tau}(x) + v_{1,\tau}(x) v_{2,\tau}(x)$$

and

$$\int_D |p_\tau(x)|\,\mathrm{d}x \leq \int_D |v_{1,\tau}(x)|\,\mathrm{d}x + \int_D |v_{2,\tau}(x)|\,\mathrm{d}x + \int_D |v_{1,\tau}(x)||v_{2,\tau}(x)|\,\mathrm{d}x$$
$$\leq |D|^{\frac{1}{2}} \|v_{1,\tau}\|_{L^2(D)} + |D|^{\frac{1}{2}} \|v_{2,\tau}\|_{L^2(D)} + \|v_{1,\tau}\|_{L^2(D)} \|v_{2,\tau}\|_{L^2(D)}$$
$$\leq CM^4 |\mathrm{Im}\,\zeta_{j,\tau}|^{-1} \leq CM^4 \tau^{-1}$$

with $C = C(k, R) > 0$. Since $|\gamma| \le \rho$, from Lemma A.2, it follows that

$$
\begin{aligned}
|(n_1 - n_2)\widehat{\ }(\gamma)| &= (2\pi)^{-\frac{3}{2}} \left| \int_D (n_1 - n_2)(x) e^{-\mathrm{i}\gamma\cdot x} \, \mathrm{d}x \right| \\
&= (2\pi)^{-\frac{3}{2}} \left| \int_D (n_1 - n_2)(x) u_{1,\tau}(x) u_{2,\tau}(x) \, \mathrm{d}x - \int_D (n_1 - n_2)(x) e^{-\mathrm{i}\gamma\cdot x} p_\tau(x) \, \mathrm{d}x \right| \\
&\le C \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)} \|u_{1,\tau}\|_{L^2(B_{2R})} \|u_{2,\tau}\|_{L^2(B_{2R})} + CM^5 \tau^{-1} \\
&\le CM^5 \left(e^{4R(\tau+\rho)} \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)} + \frac{1}{\tau} \right)
\end{aligned}
$$

for some $C = C(k, R) > 0$, which leads to the lemma. □

We now prove the following stability estimate in terms of the total fields.

Theorem A.4. Let $t > \frac{3}{2}$ and $R > 0$ be such that $D \subset B_R$. Then there exists a positive constant $C = C(R, k, t)$, which is independent of M, such that for all refractive indices n_1, n_2 satisfying (A.12) the estimate

$$
\|n_1 - n_2\|_{L^2(D)} \le CM^5 \left(-\log^- \|w_{n_1} - w_{n_2}\|_{L^2(\partial B_R \times \partial B_R)} \right)^{-\frac{t}{t+3}} \tag{A.16}
$$

holds true, where $\log^- z = \log(z)$ if $z \le 1/e$ and $\log^-(z) = -1$ if $z > 1/e$.

Remark. Using similar arguments we can obtain a stability estimate for $\|n_1 - n_2\|_{L^\infty(D)}$ by replacing (A.14) with (A.15). To simplify the presentation, we will not explore all the details here.

Proof of Theorem A.4. Without loss of generality, we assume $D \subset B_1$ and $R > 1$. In view of Fourier expansion (A.13), the estimate (A.14) and Lemma A.3, and the fact that there are less that $2\rho^3$ multi-indices $\gamma \in \mathbb{Z}^3$ with $|\gamma| \le \rho$, we have

$$
\begin{aligned}
&\|n_1 - n_2\|^2_{L^2(B_1)} \\
&\le C(k, R) M^{10} \left(2\rho^3 \left(e^{4R(\tau+\rho)} \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)} + \frac{1}{\tau} \right) \right)^2 + \frac{C(t) M^2}{\rho^{2t}} \\
&\le C(k, R, t) M^{10} \left(e^{(4R+1)(\tau+\rho)} \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)} + \frac{\rho^3}{\tau} + \frac{1}{\rho^t} \right)^2
\end{aligned}
$$

for all $\tau > \tau_0 = \sqrt{(k^2 M + 1)^2 + k^2}$ and for all $\rho \ge 2$.

We now choose $\rho = \tau^{\frac{1}{t+3}}$ and see that $\rho \ge 2$ for all $\tau \ge \tau_0 + 2^{t+3}$. Then we have

$$
\begin{aligned}
&\|n_1 - n_2\|_{L^2(B_1)} \\
&\quad \le C(k, R, t) M^5 \left(e^{(8R+2)\tau} \|S_{n_1} - S_{n_2}\|_{L^2(\partial B_R)\to L^2(\partial B_R)} + \frac{2}{\tau^{t/(t+3)}} \right) \\
&\quad \le C(k, R, t) M^5 \left(e^{(8R+2)\tau} \|w_{n_1} - w_{n_2}\|_{L^2(\partial B_R \times \partial B_R)} + \frac{2}{\tau^{t/(t+3)}} \right).
\end{aligned}
$$

Case 1. Suppose that $\|w_{n_1} - w_{n_2}\|_{L^2(\partial B_R \times \partial B_R)}$ is sufficiently small such that

$$
\tau := -\frac{1}{t+3} \frac{3}{8R+2} \log \|w_{n_1} - w_{n_2}\|_{L^2(\partial B_R \times \partial B_R)} \ge \tau_0 + 2^{t+3}, \tag{A.17}
$$

then we see that

$$\begin{aligned}\|n_1-n_2\|_{L^2(B_1)} &\le C(k,R,t)M^5\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}\\ &\le C(k,R,t)M^5\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}.\end{aligned}\tag{A.18}$$

On the other hand, (A.17) implies

$$\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\le\exp\left(-\frac{1}{3}(t+3)(8R+2)(\tau_0+2^{t+3})\right)\le e^{-1}.$$

Case 2. Otherwise, if $\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\le e^{-1}$, but (A.17) does not hold, then we see that

$$\left(\frac{(t+3)(8R+2)}{3}(\tau_0+2^{t+3})\right)^{-\frac{t}{t+3}}\le\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}.$$

Consequently, we have that

$$\begin{aligned}\|n_1-n_2\|_{L^2(B_1)} &\le \|1-n_1\|_{H^t}+\|1-n_2\|_{H^t}\le 2M\\ &\le 2M\left(\frac{(t+3)(8R+2)}{3}(\tau_0+2^{t+3})\right)^{\frac{t}{t+3}}\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}\\ &\le C(k,R,t)M\tau_0^{\frac{t}{t+3}}\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}\\ &\le C(k,R,t)M^{\frac{2t+3}{t+3}}\left(-\log\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}\right)^{-\frac{t}{t+3}}.\end{aligned}\tag{A.19}$$

Combining (A.18), (A.19), and using the a priori bounds of n_1,n_2 in the case of $\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_R\times\partial B_R)}>e^{-1}$, implies (A.16). □

We are now ready to refine [HH01, Theorem 1.2] in the following theorem.

Theorem A.5. Let $t>\frac{3}{2}$ and $0<\epsilon<\frac{t}{t+3}$. Assume that n_1,n_2 satisfy (A.12) and (A.1) also holds. Then there exists a positive constant $C=C(D,M_0,t,k,\epsilon)$, which is independent of M, such that

$$\|n_1-n_2\|_{L^2(D)}\le CM^5\left(-\log^-\|u_{n_1}^\infty-u_{n_2}^\infty\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}\right)^{-\frac{t}{t+3}+\epsilon}.$$

Proof. Without loss of generality, we may assume that $D\subset B_1$. Given any $0<\theta<1$, by following the arguments in [HH01, Theorem 1.2], one can show that there exist $\omega=\omega(k,R)>0$ and $\rho=\rho(k,R,t,M_0,\theta)>0$ with $\omega\rho>1$ such that

$$\begin{aligned}\|w_{n_1}-w_{n_2}\|_{L^2(\partial B_{2R}\times\partial B_{2R})} &\le \rho\exp\left(-\frac{1}{2}\left(-\log\frac{\|u_{n_1}^\infty-u_{n_2}^\infty\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}{\omega\rho}\right)^{\theta}\right)\\ &\le \rho\exp\left(-\frac{1}{2}\left(-\log\|u_{n_1^\infty}-u_{n_2}^\infty\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}\right)^{\theta}\right).\end{aligned}$$

It is important to remark that here we use (A.11) and follow the argument on [HH01, Page 680–681] to ensure that the constant ρ is *independent* of M (but depends on M_0).

Choosing $R = 2$ in the above inequality and plugging this inequality into (A.16) yields

$$\begin{aligned}
\|n_1 - n_2\|_{L^2(B_1)} &\le CM^5 \left(-2\log\rho + \left(-\log\|u^\infty_{n_1} - u^\infty_{n_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}\right)^\theta\right)^{-\frac{t}{t+3}} \\
&= CM^5 \left(-\log\|u_{n_1} - u_{n_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}\right)^{-\theta\frac{t}{t+3}} \left(\frac{-2\log\rho}{(-\log\|u^\infty_{n_1} - u^\infty_{n_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)})^\theta} + 1\right)^{-\frac{t}{t+3}} \\
&\le CM^5 \left(-\log\|u^\infty_{n_1} - u^\infty_{n_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}\right)^{-\theta\frac{t}{t+3}}
\end{aligned}$$

for all sufficiently small $\|u^\infty_{n_1} - u^\infty_{n_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}$. Choosing θ such that $\theta\frac{t}{t+3} = \frac{t}{t+3} - \epsilon$, the theorem can be proved as in the proof of Theorem A.4.

□

Appendix B. Example of link function

Here we want to construct a link function Φ satisfying Definition 2.1 following the idea in [NvdGW20]. Define $\varphi \in C(\mathbb{R})$ such that

$$\varphi(t) = \begin{cases} \tilde{M}_0 - t^{-a+1}, & t > t_1, \\ C_0 t + C_1, & t_2 < t < t_1, \\ (-t)^{-a+1}, & t < t_2, \end{cases}$$

where $\tilde{M}_0 > 0$, $a > 1$, $-\infty < t_2 < -1 < 1 < t_1 < \infty$, and

$$\begin{aligned}
C_0 &= \frac{\tilde{M}_0 - t_1^{-a+1} - (-t_2)^{-a+1}}{t_1 - t_2}, \\
C_1 &= (-t_2)^{-a+1} - \frac{t_2(\tilde{M}_0 - t_1^{-a+1} - (-t_2)^{-a+1})}{t_1 - t_2}.
\end{aligned}$$

Let $\psi : \mathbb{R} \to [0, \infty)$ be a smooth function such that $\mathrm{supp}\psi \subset [-1, 1]$ and $\int_{\mathbb{R}} \psi(t)dt = 1$. We now define

$$\Phi(t) := \frac{\psi * \varphi(t)}{\psi * \varphi(0)}, \quad -\infty < t < \infty,$$

and set $\tilde{M}_0 > 0$ such that

$$\frac{\tilde{M}_0}{\psi * \varphi(0)} = M_0,$$

(if necessary, we can choose $|t_1|, |t_2|$ large enough). Then, we want to verify that Φ satisfies (i)–(iii) of Definition 2.1. Indeed, by the direct computation, we have

$$\Phi'(t) = (a-1)\int_{t_1}^{\infty} \psi(t-s)s^{-a}ds + C_0\int_{t_2}^{t_1} \psi(t-s)ds + (a-1)\int_{-\infty}^{t_2} \psi(t-s)(-s)^{-a}ds,$$

which is positive for all $t \in \mathbb{R}$, and

$$\lim_{t\to\infty} \Phi(t) = M_0, \quad \lim_{t\to-\infty} \Phi(t) = 0,$$

which implies that (i) holds.

Since $0 \le \Phi(z) \le M_0$ for all z,

$$|\Phi^{(k)}(t)| \le \int_{\mathbb{R}} |\psi^{(k)}(t-s)| M_0 ds \le M_0 \int_{\mathbb{R}} |\psi^{(k)}(s)| ds < \infty,$$

and hence (ii) holds.

Finally, for $t \gg t_1$,

$$\begin{aligned}\Phi'(t) &= (a-1)\int_{-\infty}^{t-t_1} \psi(z)(t-z)^{-a}dz = (a-1)\int_{-1}^{1} \psi(z)(t-z)^{-a}dz \\ &\geq (a-1)(t+1)^{-a}\int_{-1}^{1}\psi(z)dz \approx t^{-a},\end{aligned}$$

and for $t \ll t_2$,

$$\begin{aligned}\Phi'(t) &= (a-1)\int_{t-t_2}^{\infty} \psi(z)(z-t)^{-a}dz = (a-1)\int_{-1}^{1} \psi(z)(z-t)^{-a}dz \\ &\geq (a-1)(-t-1)^{-a}\int_{-1}^{1}\psi(z)dz \approx (-t)^{-a},\end{aligned}$$

which is (iii).

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